STABILITY OF CONICAL SHOCKS IN THE THREE-DIMENSIONAL STEADY SUPERSONIC ISOTHERMAL FLOWS PAST LIPSCHITZ PERTURBED CONES*

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Abstract. We are concerned with the structural stability of conical shocks in the three-dimensional steady supersonic flows past Lipschitz perturbed cones whose vertex angles are less than the critical angle. The flows under consideration are governed by the steady isothermal Euler equations for potential flow with axisymmetry so that the equations contain a singular geometric source term. We first formulate the shock stability problem as an initial-boundary value problem with the leading conical shock-front as a free boundary, and then establish the existence and structural/asymptotic stability of global entropy solutions in BV of the problem. To achieve this, we first develop a modified Glimm scheme to construct approximate solutions via self-similar solutions as building blocks in order to incorporate with the geometric source term. Then we introduce the Glimm-type functional, based on the local interaction estimates between weak waves, the strong leading conical shock, and self-similar solutions, as well as the estimates of the center changes of the self-similar solutions. To make sure the decreasing of the Glimm-type functional, we choose appropriate weights by careful asymptotic analysis of the reflection coefficients in the interaction estimates, when the Mach number of the incoming flow is sufficiently large. Finally, we establish the existence of global entropy solutions involving a strong leading conical shock-front, besides weak waves, under the conditions that the Mach number of the incoming flow is sufficiently large and the weighted total variation of the slopes of the generating curve of the Lipschitz perturbed cone is sufficiently small. Furthermore, the entropy solution is shown to approach asymptotically the self-similar solution that is determined by the incoming flow and the asymptotic tangent of the cone boundary at infinity.

Key words. Conical shocks, structural stability, steady flow, supersonic isothermal flow, perturbed cones, Lipschitz cones, entropy solutions, BV, modified Glimm scheme, Glimm-type functional, self-similar solutions, interaction estimates, reflection coefficients, free boundary, asymptotic stability.

AMS subject classifications. 35B07, 35B20, 35D30, 76J20, 76L99, 76N10

1. Introduction.

We are concerned with the structural stability of conical shocks in the three-dimensional (3-D) steady supersonic flows past Lipschitz perturbed cones whose vertex angles are less than the critical angle. The shock stability problem of steady supersonic flows past Lipschitz cones is fundamental for the mathematical theory of multidimensional (M-D) hyperbolic systems of conservation laws, since its solutions are time-asymptotic states and global attractors of general entropy

*Submitted to the editors DATE.

Funding: The work of the first author was supported in part by the UK Engineering and Physical Sciences Research Council under Grant EP/L015811/1 and the Royal Society-Wolfson Research Merit Award (UK). The work of the second author was supported in part by NSFC Project 11801549, NSFC Project 11971024, and the Start-Up Research Grant (Project No. Y8S001104) from the Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences. The work of the third author was supported in part by NSFC Project 11421061, NSFC Project 11031001, NSFC Project 11121101, the 111 Project B08018 (China), and by the Shanghai Natural Science Foundation 15ZR1403900.

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solutions of time-dependent initial-boundary value problems (IBVP) with rich nonlinear phenomena, besides its importance to many areas of applications including aerodynamics; see [1, 5, 13, 16] and the references cited therein. As indicated in [13], when a uniform supersonic flow with constant speed from the far-field (minus infinity) hits a straight-sided symmetric cone whose vertex angle is less than the critical angle, there is a supersonic straight-sided conical shock attached to the vertex of cone, and the state between the conical shock-front and the cone can be obtained by the shooting method, which is a self-similar solution (see Fig. 1.1). In this paper, we focus our analysis on the stability of the supersonic conical shock-front, along with the background self-similar solution, in the steady supersonic Euler flows that are isothermal and symmetric with respect to the x–axis under the Lipschitz perturbation of the cones whose boundary surfaces in $\mathbb{R}^3$ are formed by the rotation of generating curves: $y = b(x)$ for $x > 0$ around the x–axis (see Fig. 1.2).

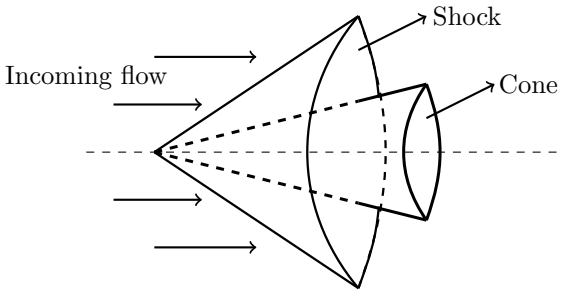

Fig. 1.1. *The strong straight-sided conical shock in the supersonic flow past a straight-sided cone*

More precisely, the governing 3-D Euler equations for steady isothermal potential conical flows are of the form:

$$\begin{cases} \partial_x(\rho u) + \partial_y(\rho v) = -\frac{\rho v}{y}, \\ \partial_x v - \partial_y u = 0, \end{cases} \tag{1.1}$$

together with the Bernoulli law:

$$\frac{u^2+v^2}{2} + c^2 \ln \rho = \frac{u_\infty^2}{2} + c^2 \ln \rho_\infty, \tag{1.2}$$

where (u, v) is the velocity in the (x, y)–coordinates, ρ is the flow density, $U_\infty = (u_\infty, 0)^\top$ and ρ_∞ are the velocity and the density of the incoming flow, respectively. The Bernoulli law (1.2) is derived from the constitutive relation for the isothermal gas between pressure p and density ρ:

$$p = c^2 \rho, \tag{1.3}$$

where constant $c > 0$ is the sound speed.

Without loss of generality, we may set $\rho_\infty = 1$; otherwise, we can simply scale: $\rho \to \frac{\rho}{\rho_\infty}$, in system (1.1)–(1.2) which is invariant in terms of the form. For fixed sound speed $c > 0$, the Mach number:

$$M_\infty = \frac{u_\infty}{c}$$

is equivalent to u_∞; in particular, the condition that the Mach number M_∞ is sufficiently large is equivalent to that the incoming velocity u_∞ is sufficiently large.

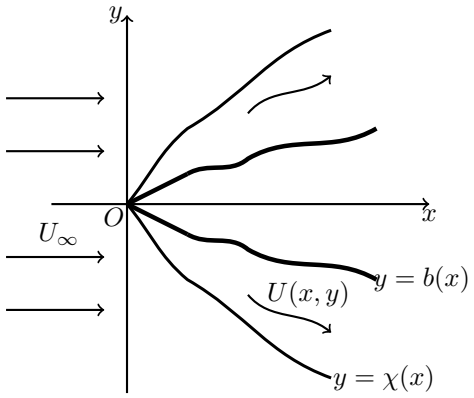

Fig. 1.2. *The strong conical shock $y = \chi(x)$ in the steady supersonic flow past a Lipschitz cone*

System (1.1) can be written in the form:

$$\partial_x W(U) + \partial_y H(U) = G(U, y) \tag{1.4}$$

with $U = (u, v)^\top$, where

$$W(U) = (\rho u, v)^\top, \quad H(U) = (\rho v, -u)^\top, \quad G(U, y) = (-\frac{\rho v}{y}, 0)^\top,$$

and ρ is a function of U through the Bernoulli law (1.2).

When $\rho > 0$ and $u > c$, U can also be represented by $W = (\rho u, v)^\top$, *i.e.*, $U = U(W)$, by the implicit function theorem, since the Jacobian:

$$\det(\nabla_U W(U)) = -\frac{\rho}{c^2}(u^2 - c^2) < 0.$$

Regarding x as the time variable, (1.4) can be written as

$$\partial_x W + \partial_y H(U(W)) = G(U(W), y). \tag{1.5}$$

Therefore, system (1.1) becomes a hyperbolic system of conservation laws with source terms of form (1.5). Such nonhomogeneous hyperbolic systems of conservation laws also arise naturally in other problems from many important applications, which exhibit rich phenomena; for example, see [5, 6, 7, 8, 13, 16] and the references cited therein.

Throughout this paper, the following conditions are assumed:

($\mathbf{H_1}$) The Lipschitz generating curve of the cone, $y = b(x) < 0$ for $x > 0$, is a small perturbation of line $y = b_0 x$, for some constant $b_0 < 0$, and satisfies

$$b(x) = b_0 x \qquad \text{for } x \in [0, x_0]$$

with some $x_0 > 0$, and

$$\|b'_+(\cdot) - b_0\|_{BV(\mathbb{R}_+)} \le \varepsilon \qquad \text{for some } \varepsilon > 0,$$

where $b'_+(x) = \lim_{y \to x+0} \frac{b(y)-b(x)}{y-x} \in BV([0, \infty))$.

($\mathbf{H_2}$) The incoming flow velocity $U_\infty = (u_\infty, 0)^\top$ is supersonic:

$$M_\infty > 1.$$

Given a perturbed generating curve $y = b(x) < 0$ of the cone, the problem is symmetric with respect to the x–axis. Thus, it suffices to consider the problem in the following domain Ω in the half-space $y \le 0$ outside the half-cone:

$$\Omega = \big\{(x, y) \,:\, x \ge 0,\, y < b(x)\big\}$$

with its boundary:

$$\partial\Omega = \big\{(x, y) \,:\, x \ge 0,\, y = b(x)\big\},$$

and the corresponding outer normal vector to $\partial\Omega$ at a differentiable point $x \in \partial\Omega$:

$$\mathbf{n} = \mathbf{n}(x, b(x)) = \frac{(-b'(x), 1)^\top}{\sqrt{1 + (b'(x))^2}}.$$

With this setup, the shock stability problem can be formulated into the following initial-boundary value problem (IBVP) for system (1.4):

Cauchy Condition:

$$U|_{x=0} = U_\infty := (u_\infty, 0)^\top, \tag{1.6}$$

Boundary Condition:

$$U \cdot \mathbf{n}\,|_{\partial\Omega} = 0. \tag{1.7}$$

We first introduce the notion of entropy solutions for problem (1.4)–(1.7).

Definition 1.1. *Consider IBVP (1.4)–(1.7) in Ω. A vector function $U(x, y) \in (BV_{\rm loc} \cap L^\infty)(\Omega)$ is an entropy solution of (1.4)–(1.7) if the following conditions are satisfied*:

(i) *For any test function $\phi \in C_0^\infty(\mathbb{R}^2)$,*

$$\int_\Omega \big\{W(U)\phi_x + H(U(W))\phi_y + G(U, y)\phi\big\}\,\mathrm{d}x\mathrm{d}y + \int_0^\infty W(U_\infty)\phi(0, y)\,\mathrm{d}y = 0; \tag{1.8}$$

(ii) *For any convex entropy pair $(\mathcal{E}, \mathcal{Q})$ with respect to W of (1.5), i.e., $\nabla^2\mathcal{E}(W) \ge 0$ and $\nabla\mathcal{Q}(W) = \nabla\mathcal{E}(W)\nabla H(U(W))$,*

$$\begin{aligned}\int_\Omega \big\{\mathcal{E}(W(U))\psi_x + \mathcal{Q}(W(U))\psi_y + \nabla_W\mathcal{E}(W(U))G(U)\psi\big\}\,\mathrm{d}x\mathrm{d}y \\ + \int_0^\infty \mathcal{E}(W(U_\infty))\psi(0, y)\,\mathrm{d}y \ge 0 \qquad \text{for any } \psi \in C_0^\infty(\mathbb{R}^2) \text{ with } \psi \ge 0.\end{aligned} \tag{1.9}$$

We now state the main results of this paper.

THEOREM 1.2 (Main theorem). *Let conditions* $(\mathbf{H_1})$–$(\mathbf{H_2})$ *hold. Assume that*

$$\int_0^\infty \big(1+|b(x)|\big)\,d\mu(x) < \varepsilon, \tag{1.10}$$

where $\mu(x) = T.V.\{b'_+(\tau) : \tau \in [0,x)\}$. *Then the following statements hold*:

(i) (*Global existence*): *If* M_∞ *is sufficiently large and* ε *is sufficiently small, IBVP* (1.4)–(1.7) *admits a global entropy solution* $U(x,y)$ *with bounded total variation*:

$$\sup_{x>0} T.V.\big\{U(x,y) : -\infty < y < b(x)\big\} < \infty, \tag{1.11}$$

in the sense of Definition 1.1. *The entropy solution* $U(x,y)$ *contains a strong leading shock-front* $y = \chi(x) = \int_0^x s(\tau)\,d\tau$ *with* $s(x) \in BV(R_+)$, *which is a small perturbation of the strong straight-sided conical shock-front* $y = s_0 x$, *and* $U(x,y)$ *between the leading shock-front and the cone surface is a small perturbation of the background self-similar solution of the straight-sided cone case, where* s_0 *denotes the slope of the corresponding straight-sided shock-front when the straight-sided cone is given by* $y = b_0 x$.

(ii) (*Asymptotic behavior*): *For the entropy solution* $U(x,y)$ *constructed in* (i),

$$\lim_{x\to\infty} \sup\big\{|U(x,y) - \varpi(\sigma_\infty; O_\infty)| : \chi(x) < y < b(x)\big\} = 0, \tag{1.12}$$

with $\varpi(\sigma_\infty; O_\infty)$ *satisfying*

$$\varpi(s_\infty; O_\infty) = \Theta(s_\infty), \quad \varpi(b'_\infty; O_\infty)\cdot(-b'_\infty, 1) = 0, \tag{1.13}$$

where

$$s_\infty = \lim_{x\to\infty} s(x), \quad b'_\infty = \lim_{x\to\infty} b'_+(x), \tag{1.14}$$

$\varpi(\sigma_\infty; O_\infty)$ *is the state of the self-similar solution with* $\sigma_\infty = \frac{y}{x - X^*_\infty}$ *and* $O_\infty = (X^*_\infty, 0)$ *as its self-similar variable and center respectively for some* X^* *determined by the asymptotic limit of* $b'_+(x)$ *as* $x \to \infty$, *and* $\Theta(s)$ *denotes the state connected to state* U_∞ *by the strong leading shock-front of speed* s.

Some efforts have been made on the shock stability problem for the perturbed cones that are small perturbations of the straight-sided cone during the last three decades. The local piecewise smooth solutions for polytropic potential flow near the cone vertex were given in [10, 11] for both symmetrically perturbed cone and pointed body, respectively. The global existence of weak solutions was first analyzed via a modified Glimm scheme by Lien-Liu [22] for the uniform supersonic isentropic Euler flow past over a piecewise straight-side cone, provided that the cone has a small opening angle, the initial strength of the shock-front is sufficiently weak, and the Mach number of the incoming flow is sufficiently large. It is further considered in Wang-Zhang [25] for supersonic potential flow for the adiabatic exponent $\gamma \in (1,3)$ over a symmetric Lipschitz cone with arbitrary opening angle that is less than the critical angle, so that a global weak solution could be constructed, which is a small perturbation of the self-similar solution under the conditions that the total variation of the slopes of the perturbed generating curves of the cone is small and the Mach number of the incoming flow is sufficiently large.

Another concern is whether global piecewise smooth solutions could be constructed when the surface of the perturbed cone is smooth. Using the weighted energy

methods, the global existence of piecewise smooth solutions was established in Chen-Xin-Yin [12] for the 3-D axisymmetric potential flow past a symmetrically perturbed cone under the assumptions that the attached angle is sufficiently small and the Mach number of the incoming flow is sufficiently large. This result was also extended to the M-D potential flow case (see [21] for more details). In [27], the global existence of the M-D conical shock solutions was established when the uniform supersonic incoming flow with large Mach number past a generally curved sharp cone under a certain boundary condition on the cone surface. On the other hand, by using the delicate expansion of the background solution, the global existence and stability of a steady conical shock wave was established in Cui-Yin [14, 15] for the symmetrically perturbed supersonic flow past an infinitely long conic body, when the the vertex angle is less than the critical angle. More recently, by constructing new background solutions that allow the incoming flows to tend to the speed limit, the global existence of steady symmetrically conical shock solutions was established in Hu-Zhang [19] when a supersonic incoming potential flow hits a symmetrically perturbed cone with the opening angle less than the critical angle. We also remark that some important results have been obtained on the stability of M-D transonic shocks under symmetric perturbations of the straight-sided cones or the straight-sided wedges, as well as on Radon measure solutions for steady compressible Euler equations of hypersonic-limit conical flows; see [2, 3, 4, 23, 28] and the references cited therein.

In this paper, we establish the global existence and structural/asymptotic stability of conical shock-front solutions in BV in the flow direction when the isothermal flows (*i.e.*, $\gamma = 1$) past Lipschitz perturbed cones that are small perturbations of the straight-sided one. Mathematically, our problem can be formulated as a free boundary problem governed by 2-D steady isentropic irrotational Euler flows with geometric structure. There are two difficulties for solving this problem: One is the singularity generated by the geometric source term, and the other is that, for our case $\gamma = 1$, the two genuinely nonlinear characteristics are superposed into a degenerate one when the Mach number of the incoming flow tends to infinity, which is delicate to handle in the construction of approximate solutions.

To overcome these obstacles and make sure the non-increasing of the ongoing designed Glimm-type functional, we first develop a modified Glimm scheme to construct approximate solutions $U_{\Delta x,\vartheta}(x,y)$ via the self-similar solutions as building blocks in order to incorporate with the geometric source term. To achieve this, we make careful asymptotic expansions of the self-similar solutions up to second order with respect to M_∞^{-1}. In addition to the shock waves and rarefaction waves generated by solving the Riemann problem, there is another new type of discontinuities generated by the center changes and the corresponding updated self-similar variables of the self-similar solutions, owing to the Lipschitz perturbation of the cone. In order to deal with this new discontinuity, we introduce new functionals $L_{\rm c}$, $Q^{(1)}_{\rm wc}$, $Q^{(2)}_{\rm wc}$, and $Q_{\rm ce}$ (see Definitions 7.2–7.3) in the construction of the Glimm-type functional to control the center changes. Finally, in order to ensure the decreasing of the Glimm-type functional, we make more precise asymptotic expansion analysis of the background solutions with respect to the Mach number M_∞ of the incoming flow, obtain their expansion formulas when M_∞ sufficiently large, and then make full use of the reflection coefficients $K_{\rm r}, K_{\rm w}, K_{\rm s}$, and $\mu_{\rm w}$ of the weak waves reflected from both the boundary and the strong leading shock, and the self-similar solutions reflected from the strong leading shock to derive that

$$|K_{\rm r}|\big(|K_{\rm w}|+|K_{\rm s}||\mu_{\rm w}|\big)=1-(8b_0^4+2b_0^2+1)m_0^{-1}M_\infty^{-1}+O(M_\infty^{-2})+O(e^{-m_0M_\infty^2}),$$

which is strictly less than 1 when M_∞ is sufficiently large, where $m_0 = \frac{b_0^2}{2(1+b_0^2)} > 0$. We do this expansion with respect to sufficiently large M_∞ in order to overcome the superposed singularity caused for the case that $\gamma = 1$. Based on this, we can choose some appropriate weights, independent of M_∞, in the construction of the Glimm-type functional and then show that the functional is monotonically decreasing. With these, the convergence of the approximate solutions and the existence of an entropy solution are followed by the standard approach for the Glimm-type scheme as in [17, 20]; see also [9, 16, 24].

For the asymptotic behavior of the entropy solution, we need further estimates on the approximation solutions $U_{\Delta x,\vartheta}(x,y)$. The key point here is that a new term $\mathcal{C}_{\Delta x,\vartheta}(x)$ is introduced to measure the total variation for the changes of centers $X^*_{\Delta x,\vartheta}$ in $U_{\Delta x,\vartheta}(x,y)$ and show that this term eventually approaches to zero by further estimates on the approximate solutions, so that $X^*_{\Delta x,\vartheta}$ tends to a constant X^*_ϑ for $\Delta x \to 0$. This is different from the wedge case that has been handled in [9, 29]. In addition, we also prove that the total variation of the weak waves also approaches to zero as $x \to \infty$. Then, by employing the Glimm-Lax theory [18], we obtain the asymptotic behavior of the entropy solution that tends to a self-similar solution with $X^*_\infty = \lim_{x\to\infty} X^*_\vartheta(x, b(x))$ as its center.

The rest of this paper is organized as follows: In §2, we recall some basic facts for the homogeneous system of (1.1), which are required for subsequent developments. In §3, we analyze the background solutions for steady supersonic flows past the unperturbed straight-sided cones and obtain some detailed asymptotic estimates for the self-similar solutions as $M_\infty \to \infty$. In §4, we solve two types of Riemann problems, while a modified Glimm scheme is developed for the construction of approximate solutions in §5. The local wave interaction estimates are given in §6 for large M_∞. In §7, we construct the Glimm-type functional and prove its monotonicity that leads to the existence theory by following the standard procedure of [17, 20]; see also [9, 16, 24]. In §8, we analyze the asymptotic behavior of the entropy solutions. Finally, in Appendix A, we give a detail proof of Lemma 2.1.

2. Homogeneous System. In this section, we present some basic properties of the homogeneous system of (1.1), *i.e.*, $G(U,y) \equiv 0$. For this case, system (1.4) can be reduced to the following conservation form:

$$\partial_x W(U) + \partial_y H(U) = 0. \tag{2.1}$$

For $u > c$, system (2.1) is strictly hyperbolic and has two distinct eigenvalues:

$$\lambda_j(U) = \frac{uv + (-1)^j c\sqrt{u^2+v^2-c^2}}{u^2-c^2} \qquad \text{for } j = 1, 2,$$

and the corresponding two right-eigenvectors:

$$r_j(U) = e_j(U)(-\lambda_j(U), 1)^\top \qquad \text{for } j = 1, 2,$$

where $e_j(U) > 0$ can be chosen so that $r_j(U) \cdot \nabla_U \lambda_j(U) \equiv 1$ for $j = 1, 2$.

The fact that $e_j(U) > 0, j = 1, 2$, is a consequence of the following lemma whose proof is given in Appendix A.

LEMMA 2.1. *If $\lambda_j(U)$ is the j-th eigenvalue of (2.1) and $r_j(U)$ is the corresponding eigenvector satisfying $r_j(U) \cdot \nabla_U \lambda_j(U) \equiv 1$ for $u > c$ for $j = 1, 2$. Then*

$$e_j(U) = \frac{\sqrt{M^2-1}}{c^2 M^6}\big(u\sqrt{M^2-1} + (-1)^{j+1} v\big)^3 > 0 \qquad \text{for } j = 1, 2, \tag{2.2}$$

where $M = \frac{q}{c}$ is the Mach number and $q = \sqrt{u^2+v^2}$ is the fluid speed.

3. Properties of the Background Solutions. In this section, we study the conical flow past a straight-sided cone, *i.e.*, $b(x) = b_0 x$ for $x \geq 0$. According to [13], problem (1.4)–(1.7) admits a self-similar solution $(u(\sigma), v(\sigma), \rho(\sigma))$ with $\sigma = \frac{y}{x}$ as its self-similar variable for this case. Then it can be reduced to a boundary value problem of an ordinary differential equation, whose solution consists of a straight-sided conical shock-front issuing from the cone vortex, when $|b_0|$ is less than the critical angle (see Fig. 3.1).

Let $y = s_0 x$ be the location of the shock-front. Then problem (1.1)–(1.6) (with $\rho_\infty = 1$ by scaling) becomes

$$
\begin{cases}
(\sigma u - v)\rho_\sigma + \sigma\rho u_\sigma - \rho v_\sigma = \frac{\rho v}{\sigma}, & s_0 < \sigma < b_0,\\
u_\sigma + \sigma v_\sigma = 0, & s_0 < \sigma < b_0,\\
\frac{c}{\rho}\rho_\sigma + u u_\sigma + v v_\sigma = 0, & s_0 < \sigma < b_0,\\
\rho(u s_0 - v) = u_\infty s_0, & \sigma = s_0,\\
u + v s_0 = u_\infty, & \sigma = s_0,\\
v - u b_0 = 0, & \sigma = b_0,
\end{cases}
\tag{3.1}
$$

and

$$
\big(u(\sigma), v(\sigma)\big) = (u_\infty, 0), \quad \sigma < s_0. \tag{3.2}
$$

System $(3.1)_1$–$(3.1)_3$ can also be rewritten in an equivalent form as

$$
\begin{cases}
u_\sigma = \frac{c^2 v}{(1+\sigma^2)c^2 - (v-\sigma u)^2},\\
v_\sigma = -\frac{c^2 v}{\sigma((1+\sigma^2)c^2 - (v-\sigma u)^2)},\\
\rho_\sigma = \frac{\rho v(v-\sigma u)}{\sigma((1+\sigma^2)c^2 - (v-\sigma u)^2)}.
\end{cases}
\tag{3.3}
$$

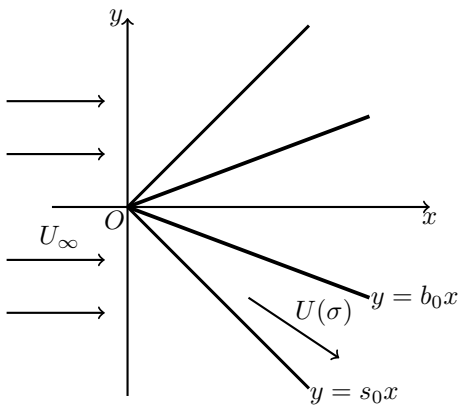

Fig. 3.1. *Steady supersonic flow past an unperturbed straight-sided cone*

To study the self-similar solution, we need some properties of the shock polar.

LEMMA 3.1. *Let $b_+ < 0$. Then there exist constants $K' > 0$ and $K'' \in (0, K')$ independent of M_∞ such that, for M_∞ sufficiently large, the following system of equations*:

$$\rho_+(u_+s_+ - v_+) = u_\infty s_+, \tag{3.4}$$

$$u_+ + v_+s_+ = u_\infty, \tag{3.5}$$

$$u_+b_+ - v_+ = 0, \tag{3.6}$$

$$\frac{u_+^2 + v_+^2}{2} + c^2 \ln \rho_+ = \frac{u_\infty^2}{2}, \tag{3.7}$$

has a unique solution (u_+, v_+, ρ_+, s_+) with

$$s_+ \in \big(b_+ - K'e^{-m_+M_\infty^2},\ b_+ - K''e^{-m_+M_\infty^2}\big) \qquad \text{for } m_+ := \tfrac{b_+^2}{2(1+b_+^2)}. \tag{3.8}$$

In addition,

$$u_+ = \Big(\frac{1}{1+b_+^2} + O(1)e^{-m_+M_\infty^2}\Big)u_\infty, \tag{3.9}$$

$$v_+ = \Big(\frac{b_+}{1+b_+^2} + O(1)e^{-m_+M_\infty^2}\Big)u_\infty, \tag{3.10}$$

$$\rho_+ = \exp\big\{m_+M_\infty^2\big(1 + O(1)e^{-m_+M_\infty^2}\big)\big\}, \tag{3.11}$$

where $O(1)$ is independent of M_∞.

Proof. We divide the proof into three steps.

1. Equations (3.5)–(3.6) yield

$$u_+ = \frac{u_\infty}{1+b_+s_+}, \qquad v_+ = \frac{b_+u_\infty}{1+b_+s_+}, \tag{3.12}$$

which implies that $u_+ > c$ for sufficiently large M_∞. Using (3.4), we have

$$\rho_+ = \frac{s_+(1+s_+b_+)}{s_+ - b_+}. \tag{3.13}$$

Then substituting (3.12)–(3.13) into (3.7) leads to

$$\frac{1}{2}\Big(\frac{1+b_+^2}{(1+b_+s_+)^2} - 1\Big) + \ln\Big(\frac{s_+(1+b_+s_+)}{s_+ - b_+}\Big)M_\infty^{-2} = 0. \tag{3.14}$$

2. In order to solve equation (3.14), we define

$$\varphi(s) := \frac{1}{2}\Big(\frac{1+b_+^2}{(1+b_+s)^2} - 1\Big) + \ln\Big(\frac{s(1+b_+s)}{s - b_+}\Big)M_\infty^{-2} \qquad \text{for } s < b_+. \tag{3.15}$$

Since $b_+ < 0$, it is direct to verify that

$$\lim_{s\to b_+-} \varphi(s) = \lim_{s\to-\infty} \varphi(s) = \infty. \tag{3.16}$$

In addition, for $K > 0$,

$$\varphi(b_+ - Ke^{-m_+M_\infty^2}) = h(K, M_\infty), \tag{3.17}$$

where

(3.18)
$$h(K,M_\infty)=\frac{1}{2}\Big(\frac{1+b_+^2}{(1+b_+^2-b_+Ke^{-m_+M_\infty^2})^2}-1\Big)+\frac{b_+^2}{2(1+b_+^2)}$$
$$+\big(\ln|b_+-Ke^{-m_+M_\infty^2}|+\ln(1+b_+^2-b_+Ke^{-m_+M_\infty^2})-\ln K\big)M_\infty^{-2}.$$

Note that, for $K>0$,

$$h(K,M_\infty)<\big(\ln|b_+-Ke^{-m_+M_\infty^2}|+\ln\big(1+b_+^2-b_+Ke^{-m_+M_\infty^2}\big)-\ln K\big)M_\infty^{-2}.$$

Then, for $M_\infty>\frac{\sqrt{\left|\ln(1+2b_+^2-2\sqrt{b_+^2(b_+^2+1)})\right|}}{\sqrt{m_+}}$, we can choose appropriate $K'>0$ such that

$$h(K',M_\infty)<-\frac{m_+}{\ln(1+2b_+^2-2\sqrt{b_+^2(b_+^2+1)})}$$
$$\times\Big\{\ln\big((1+2b_+^2-2\sqrt{b_+^2(b_+^2+1)})K'-b_+\big)$$
$$+\ln\big(1+b_+^2-(1+2b_+^2-2\sqrt{b_+^2(b_+^2+1)})b_+K'\big)-\ln K'\Big\}<0.$$

On the other hand, since

$$\lim_{K\to0+}\ln K=-\infty,$$

then, for M_∞ sufficiently large, we can also choose another constant $K''\in(0,K')$ so that

$$h(K'',M_\infty)>0.$$

These lead to

$$\varphi(b_+-K'e^{-m_+M_\infty^2})<0,\qquad \varphi(b_+-K''e^{-m_+M_\infty^2})>0,$$

which implies that $\varphi(s)=0$ has two solutions that lie in $(-\infty,\ b_+-K'e^{-m_+M_\infty^2})$ and $(b_+-K'e^{-m_+M_\infty^2},\ b_+-K''e^{-m_+M_\infty^2})$, respectively.

3. The properties of the shock polar indicate that $\varphi(s)=0$ has at most two solutions in $(-\infty,b_+)$. Therefore, $\varphi(s)=0$ has a unique solution in $(b_+-K'e^{-m_+M_\infty^2},\ b_+-K''e^{-m_+M_\infty^2})$, which gives the uniqueness of (u_+,v_+,ρ_+,s_+) and

$$s_+\in(b_+-K'e^{-m_+M_\infty^2},\ b_+-K''e^{-m_+M_\infty^2}).$$

Then, by (3.12)–(3.13), we obtain the desire estimates (3.10)–(3.11). ∎

Denote

$$\varphi(s,b):=\frac{1}{2}\Big(\frac{1+b^2}{(1+bs)^2}-1\Big)+\ln(\frac{s(1+bs)}{s-b})M_\infty^{-2}.$$

LEMMA 3.2. *For M_∞ sufficiently large and $s\in[5b_0,b_0]$, $\varphi(s,b)=0$ has a unique solution $b=b(s)$ with $b(s)\in(s,0)$. Moreover,*

$$b(s)=s+O(1)e^{-m_0M_\infty^2},$$

where $m_0=\frac{b_0^2}{2(1+b_0^2)}$, and $O(1)$ depends only on $b_0<0$ but independent of M_∞.

Proof. We differentiate $\varphi(s,b)$ with respect to b to obtain

$$\frac{\partial\varphi(s,b)}{\partial b}=\frac{(b-s)^2+(1+s^2)(1+bs)^2M_\infty^{-2}}{(b-s)(1+bs)^3}.$$

To estimate the zero points of $\frac{\partial\varphi(s,b)}{\partial b}$ in b, let

$$\varphi_1:=(b-s)^2+(1+s^2)(1+bs)^2M_\infty^{-2}.$$

Then

$$\frac{\partial\varphi_1}{\partial b}=2\big(b-s+(1+s^2)(1+bs)sM_\infty^{-2}\big).$$

For sufficiently large M_∞, $\frac{\partial\varphi_1}{\partial b}>0$. Thus, $\varphi_1=0$ has at most one solution in $(s,0)$, which implies that $\frac{\partial\varphi(s,b)}{\partial b}=0$ has at most one zero point in $(s,0)$. On the other hand, by a direct computation, we have

$$\lim_{b\to s+}\varphi(s,b)=\infty,$$

and

$$\begin{aligned}\varphi(s,s+Ke^{-m_0M_\infty^2})&=\frac{1}{2}\Big(\frac{1+(s+Ke^{-m_0M_\infty^2})^2}{(1+s^2+sKe^{-m_0M_\infty^2})^2}-1\Big)+\frac{b_0^2}{2(1+b_0^2)}\\&\quad+\big(\ln|s|+\ln(1+s^2+sKe^{-m_0M_\infty^2})-\ln K\big)M_\infty^{-2}\\&<\frac{b_0^2}{2(1+b_0^2)}-\frac{s^2}{2(1+s^2)}+\big(\ln|s|+\ln(1+s^2)-\ln K\big)M_\infty^{-2}<0\end{aligned}$$

for appropriate $K>0$ and sufficiently large M_∞, which imply the existence of $b(s)$.□

LEMMA 3.3. *Let $(u(s_0),v(s_0),\rho(s_0))$ be a state on the shock polar passing through $(u_\infty,0,\rho_\infty)$ with speed s_0. Then the following two statements are equivalent*:

(i) *The density increases across the shock in the flow direction*:

$$\rho(s_0)>1=\rho_\infty. \tag{3.19}$$

(ii) *The shock speed s_0 must be between $\lambda_1(s_0)$ and $\lambda_1(U_\infty)$*:

$$\lambda_1(s_0)<s_0<\lambda_2(s_0),\qquad s_0<\lambda_1(U_\infty), \tag{3.20}$$

where

$$\lambda_j(s_0)=\frac{u(s_0)v(s_0)+(-1)^jc\sqrt{u^2(s_0)+v^2(s_0)-c^2}}{u^2(s_0)-c^2}\qquad \text{for } j=1,2.$$

Proof. We divide the proof into two steps.

1. Case (i) ⇒ (ii). By the Bernoulli law and the Rankine-Hugoniot relation in (3.1), we have

$$\frac{\alpha^2}{\alpha^2-1}\ln\alpha=\frac{s_0^2}{2(1+s_0^2)}M_\infty^2,\quad \frac{\ln\alpha}{\alpha^2-1}=\frac{\big(v(s_0)-s_0u(s_0)\big)^2}{2c^2(1+s_0^2)}\qquad \text{for } \alpha:=\rho(s_0).$$

Denote

$$f(\alpha) := \frac{\alpha^2}{\alpha^2-1}\ln\alpha \qquad \text{for } \alpha > 1.$$

Then

$$f'(\alpha) = \frac{\alpha g(\alpha)}{(\alpha^2-1)^2} \qquad \text{for } \alpha > 1,$$

where $g(\alpha) = \alpha^2 - 2\ln\alpha - 1$. Since $g'(\alpha)|_{\{\alpha>1\}} > 0$, then $g(\alpha) > g(1) = 0$ for any $\alpha > 1$, which implies that

$$f'(\alpha)\big|_{\{\alpha>1\}} > 0.$$

Then

$$\frac{s_0^2}{2(1+s_0^2)}M_\infty^2 = f(\alpha) > \lim_{\alpha\to 1^+} f(\alpha).$$

Applying L'Hôspital's rule gives that $\lim_{\alpha\to 1^+} f(\alpha) = \frac{1}{2}$. Therefore, we have

$$\frac{s_0^2}{2(1+s_0^2)}M_\infty^2 > \frac{1}{2},$$

which yields that

$$s_0 < -\frac{1}{\sqrt{M_\infty^2-1}} = \lambda_1(U_\infty).$$

In the same way, we can show that, for $\alpha = \rho(s_0) > 1$,

$$\frac{\ln\alpha}{\alpha^2-1} = \frac{\big(v(s_0) - s_0u(s_0)\big)^2}{2c^2(1+s_0^2)} < \frac{1}{2},$$

which implies that $s_0 \in (\lambda_1(s_0), \lambda_2(s_0))$.

2. Case (ii) $\Rightarrow$ (i). On the contrary, assume that $\rho(s_0) \le 1 = \rho_\infty$. Then

$$f'(\alpha)\big|_{\{0<\alpha\le 1\}} \le 0,$$

so that $f(\alpha) \le \lim_{\alpha\to 1^-} f(\alpha)$ for $\alpha \in (0,1)$. It follows that

$$\frac{s_0^2}{2(1+s_0^2)}M_\infty^2 < \frac{1}{2},$$

that is,

$$s_0 > -\frac{1}{\sqrt{M_\infty^2-1}},$$

which contradicts (3.20). The proof is complete. ∎

Let $\Theta(s) = (\tilde{u}(s), \tilde{v}(s))$ be the states on the parameterized shock polar of $S_1^-(U_\infty)$ as defined in §4.2. Then we have the following lemma:

LEMMA 3.4. *For $s < \lambda_1(U_\infty)$, then $\frac{\tilde{v}(s)}{\tilde{u}(s)}$ is a strictly monotone increasing function with respect to s.*

Proof. From Lemma 3.1, we know that there is only one intersection point between the straight line $v = bu$ with $b < 0$ and the shock polar $S_1^-(U_\infty)$ in the supersonic region. This implies that the flow angle $\theta(s) = \arctan(\frac{\tilde{v}(s)}{\tilde{u}(s)})$ is a strictly monotone function of s. Furthermore, from the properties of the shock polar $S_1^-(U_\infty)$ (or see [13, 29] for more details), we also see that $\theta(s) < 0 = \theta(\lambda_1(U_\infty))$ for $s < \lambda_1(U_\infty)$.□

Now we consider the conical flows. We recall some properties of the apple curves in [13]. Given a constant state (u_1^0, v_1^0) on the shock polar through state $(u_\infty, 0)$. Let $(u_1(\sigma), v_1(\sigma))$ be the solution of equations $(3.3)_1$–$(3.3)_2$ with initial data

$$(u_1, v_1)\big|_{\sigma=\sigma_0} = (u_1^0, v_1^0) \qquad \text{with } \sigma_0 = \tfrac{u_\infty - u_1^0}{v_1^0}.$$

Then we can continue the solution, $(u_1(\sigma), v_1(\sigma))$, till the end point $(u_1(\sigma_e), v_1(\sigma_e))$ so that $\frac{v_1(\sigma_e)}{u(\sigma_e)} = \sigma_e$. The collection of end states forms an apple curve through $(u_\infty, 0)$. The solution, $(u(\sigma), v(\sigma))$, of $(3.3)_1$–$(3.3)_2$ can be found by the shooting method (see [13] for more details). Therefore, we see that

$$v(b_0) - u(b_0)b_0 = 0, \qquad (v(\sigma) - \sigma u(\sigma))\big|_{\{s_0<\sigma<b_0\}} \neq 0. \tag{3.21}$$

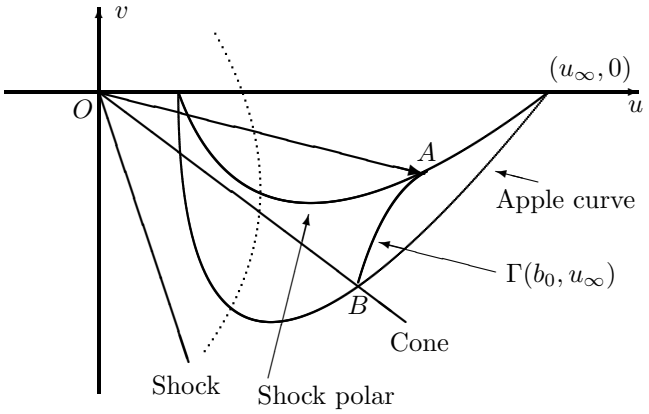

Fig. 3.2. *The apple curve and the shock polar for the self-similar solutions*

Indeed, we have

LEMMA 3.5. *For state $(u(s_0), v(s_0), \rho(s_0))$ on the shock polar through $(u_\infty, 0, \rho_\infty)$ with speed s_0,*

$$u(s_0) > 0, \quad v(s_0) < 0, \quad v(s_0) - s_0 u(s_0) > 0, \tag{3.22}$$

so that

$$v(\sigma) - \sigma u(\sigma) > 0 \qquad \text{for } s_0 < \sigma < b_0. \tag{3.23}$$

Proof. Using the Rankine-Hugoniot relations for (3.1) and Lemma 3.2 and noting that $s_0 < 0$, we have

$$u(s_0) = \frac{1}{1+s_0^2}\Big(1+\frac{s_0^2}{\rho(s_0)}\Big)u_\infty > 0,$$
$$v(s_0) = \frac{s_0}{1+s_0^2}\Big(1-\frac{1}{\rho(s_0)}\Big)u_\infty < 0,$$
$$v(s_0) - s_0u(s_0) = -\frac{s_0u_\infty}{\rho(s_0)} > 0.$$

Hence, by (3.21), we obtain (3.23) for $s_0 < \sigma < b_0$. ∎

Now we state some properties about the self-similar solutions of problem (3.1)–(3.2).

LEMMA 3.6. *For $\sigma \in (s_0, b_0)$, the free boundary problem (3.1)–(3.2) admits a unique solution $(u(\sigma), v(\sigma), \rho(\sigma))$ that satisfies the following properties*:

$$c^2(1+\sigma^2) - \big(v(\sigma)-\sigma u(\sigma)\big)^2 > 0, \tag{3.24}$$

$$v(\sigma) < 0, \qquad \rho(\sigma) > 0, \tag{3.25}$$

$$u_\sigma(\sigma) < 0, \qquad v_\sigma(\sigma) < 0, \qquad \rho_\sigma(\sigma) > 0. \tag{3.26}$$

Proof. We divide the proof into four steps.

1. Lemma 3.1 implies that the straight line $v = bu$ intersects the shock polar through $(u_\infty, 0)$. Then, from the structure of the apple curve given in [13], problem (3.1)–(3.2) has a unique solution $(u(\sigma), v(\sigma), \rho(\sigma))$.

2. We now prove (3.24)–(3.26). Define

$$\sigma_* := \sup\Big\{\sigma_0 \,:\, 0 < \frac{v(\sigma)-\sigma u(\sigma)}{\sqrt{1+\sigma^2}} < c,\ v(\sigma) < 0,\ \sigma \in [s_0, \sigma_0]\Big\}.$$

By Lemmas 3.3–3.5, we have

$$0 < \frac{v(s_0)-s_0u(s_0)}{\sqrt{1+s_0^2}} < c, \qquad v(s_0) < 0.$$

Therefore, $\sigma_* \geq s_0$.

3. We now prove $\sigma_* \geq b_0$. On the contrary, assume that $\sigma_* < b_0$. Then

$$\Big(\frac{v(\sigma_*)-\sigma_*u(\sigma_*)}{\sqrt{1+\sigma_*^2}} - c\Big)v(\sigma_*) = 0,$$
$$v(\sigma)-\sigma u(\sigma) > 0, \quad 0 < \frac{v(\sigma)-\sigma u(\sigma)}{\sqrt{1+\sigma^2}} < c \qquad \text{for } \sigma \in [s_0, \sigma_*).$$

By (3.3), we have

$$u_\sigma(\sigma) < 0, \quad v_\sigma(\sigma) < 0, \quad \rho_\sigma(\sigma) > 0 \qquad \text{for } \sigma \in [s_0, \sigma_*).$$

Denote

$$h(\sigma) := \frac{v(\sigma)-\sigma u(\sigma)}{\sqrt{1+\sigma^2}}.$$

Then

$$h'(\sigma)=\frac{v_\sigma(\sigma)-\sigma u_\sigma(\sigma)}{\sqrt{1+\sigma^2}}-\frac{u(\sigma)+\sigma v(\sigma)}{(1+\sigma^2)^{\frac{3}{2}}}<0 \qquad \text{for } \sigma\in[s_0,\sigma_*),$$

which implies that

$$\frac{v(\sigma_*)-\sigma_* u(\sigma_*)}{\sqrt{1+\sigma_*^2}}<\frac{v(s_0)-s_0u(s_0)}{\sqrt{1+s_0^2}}<c\,v(\sigma_*)<v(s_0)<0.$$

This leads to a contradiction to (3.27). Thus, $\sigma_*=b_0$.

4. From equations (3.1)–(3.2), we have

$$\rho(\sigma)>0,\quad u_\sigma(\sigma)<0,\quad v_\sigma(\sigma)<0,\quad \rho_\sigma(\sigma)>0.$$

This completes the proof. ∎

LEMMA 3.7. *For sufficiently large M_∞, solution $\big(u(\sigma),v(\sigma),\rho(\sigma)\big)$ of (3.1)–(3.2) satisfies*

$$s_0=b_0+O(1)e^{-m_0M_\infty^2}, \tag{3.27}$$

$$u(\sigma)=\Big(\frac{1}{1+b_0^2}+O(1)e^{-m_0M_\infty^2}\Big)u_\infty, \tag{3.28}$$

$$v(\sigma)=\Big(\frac{b_0}{1+b_0^2}+O(1)e^{-m_0M_\infty^2}\Big)u_\infty, \tag{3.29}$$

$$\rho(\sigma)=\exp\big\{m_0M_\infty^2\big(1+O(1)e^{-2m_0M_\infty^2}\big)\big\}, \tag{3.30}$$

for $\sigma\in[s_0,b_0]$, where $m_0:=\frac{b_0^2}{2(1+b_0^2)}$, and the bound of $O(1)$ is independent of M_∞. In particular, the largeness of M_∞ implies that

$$u(\sigma)>c \qquad \text{for any } \sigma\in(s_0,b_0). \tag{3.31}$$

Proof. We first prove (3.28). To do this, for given $b_0<0$, consider problem (3.4)–(3.7) of the planar shock polar solution with $b_+=b_0$. Then $m_+=m_0$. By Lemma 3.1, we know that solution s_+ of problem (3.4)–(3.7) satisfies

$$s_+>b_0-\tilde{K}'e^{-m_0M_\infty^2}, \tag{3.32}$$

where $\tilde{K}'>0$ independent of M_∞.

In order to obtain the conical shock with slope s_0. We set

$$b_1=\frac{v(s_0)}{u(s_0)}.$$

Then, by Lemma 3.5, we have

$$b_0u(s_0)\le b_0u(b_0)=v(b_0)\le v(s_0)=b_1u(s_0),$$

which leads to

$$b_0\le b_1\le 0.$$

By Lemma 3.4, we can further deduce that

$$s_+ < s_0. \tag{3.33}$$

Since $s_0 < b_0$, then we obtain by estimates (3.32)–(3.33) that

$$s_0 = b_0 + O(1)e^{-m_0 M_\infty^2}. \tag{3.34}$$

On the other hand, in the same way as in the proof of Lemma 3.1, we can prove

$$\varphi(s_0, b_1) = 0.$$

Then, by Lemma 3.2, we have

$$b_1 = s_0 + O(1)e^{-m_0 M_\infty^2} = b_0 + O(1)e^{-m_0 M_\infty^2}. \tag{3.35}$$

Since $(u(s_0), v(s_0))$ solves the equations:

$$u(s_0) + s_0 v(s_0) = u_\infty, \qquad u(s_0)b_1 - v(s_0) = 0,$$

then, employing estimates (3.34)–(3.35), we have

$$\begin{aligned} u(s_0) &= \frac{1}{1 + b_1 s_0} = \Big(\frac{1}{1+b_0^2} + O(1)e^{-m_0 M_\infty^2}\Big)u_\infty, \\ v(s_0) &= \frac{b_1}{1 + b_1 s_0}\Big(\frac{b_0}{1+b_0^2} + O(1)e^{-m_0 M_\infty^2}\Big)u_\infty. \end{aligned}$$

Therefore, using the monotonicity of $(u(\sigma), v(\sigma))$ again that

$$u(s_0)b_0 \le u(\sigma)b_0 \le u(b_0)b_0 = v(b_0) \le v(\sigma) \le v(s_0),$$

we derive estimates (3.28)–(3.29). Finally, by the Bernoulli law, together with the estimates of $(u(\sigma), v(\sigma))$, we can obtain (3.30). Moreover, for M_∞ sufficiently large, by (3.28), we can obtain (3.31). ∎

LEMMA 3.8. *For M_∞ sufficiently large, the following asymptotic expansions hold: For any $\sigma \in [s_0, b_0]$,*

$$\lambda_1(\sigma) = b_0 - (1+b_0^2)^{\frac{3}{2}} M_\infty^{-1} + O(1)M_\infty^{-2} + O(1)e^{-m_0 M_\infty^2}, \tag{3.36}$$

$$\lambda_2(\sigma) = b_0 + (1+b_0^2)^{\frac{3}{2}} M_\infty^{-1} + O(1)M_\infty^{-2} + O(1)e^{-m_0 M_\infty^2}, \tag{3.37}$$

$$\frac{e_1(\sigma)}{u_\infty} = \frac{1}{(1+b_0^2)^2} + 3b_0(1+b_0^2)^{-\frac{3}{2}} M_\infty^{-1} + O(1)M_\infty^{-2} + O(1)e^{-m_0 M_\infty^2}, \tag{3.38}$$

$$\frac{e_2(\sigma)}{u_\infty} = \frac{1}{(1+b_0^2)^2} - 3b_0(1+b_0^2)^{-\frac{3}{2}} M_\infty^{-1} + O(1)M_\infty^{-2} + O(1)e^{-m_0 M_\infty^2}, \tag{3.39}$$

$$\frac{e_1(\sigma)}{e_2(\sigma)} = 1 + 6b_0(1+b_0^2)^{\frac{1}{2}}\, M_\infty^{-1} + O(1)M_\infty^{-2} + O(1)e^{-m_0 M_\infty^2}, \tag{3.40}$$

where $e_j(\sigma) = e_j(U(\sigma))$, $j = 1, 2$, and the universal bound of $O(1)$ is independent of M_∞.

Proof. By Lemma 3.5 and Taylor's formula when M_∞ is large enough, we know that

$$\begin{aligned}\lambda_1(\sigma) &= \frac{\frac{v(\sigma)}{u(\sigma)} - \frac{1}{\sqrt{M^2-1}}}{1+\frac{v(\sigma)}{u(\sigma)}\frac{1}{\sqrt{M^2-1}}} = \frac{b_0 - \sqrt{1+b_0^2}M_\infty^{-1} + O(1)e^{-m_0M_\infty^2}}{1+b_0\sqrt{1+b_0^2}M_\infty^{-1}+O(1)e^{-m_0M_\infty^2}}\\ &= b_0 - (1+b_0^2)^{\frac{3}{2}}M_\infty^{-1} + O(1)M_\infty^{-2} + O(1)e^{-m_0M_\infty^2}.\end{aligned}$$

The proof of (3.37) is entirely similar.

Next, for $e_j(\sigma)$ in (2.2) with $j=1,2$, using the same method again when M_∞ is large enough, we have

$$\begin{aligned}\frac{e_1(\sigma)}{u_\infty} &= \sqrt{\frac{M^2}{M_\infty^2}-\frac{1}{M_\infty^2}}\left(\frac{u}{q}\sqrt{1-\frac{1}{M^2}}+\frac{v}{q}\frac{1}{M}\right)^3\\ &= \sqrt{\frac{1}{1+b_0^2}-\frac{1}{M_\infty^2}}\left(\sqrt{\frac{1}{1+b_0^2}}\sqrt{1-\frac{(1+b_0^2)}{M_\infty^2}}+\frac{b_0}{M_\infty}\right)^3+O(1)e^{-m_0M_\infty^2}\\ &= \frac{1}{(1+b_0^2)^2}+3b_0(1+b_0^2)^{-\frac{3}{2}}M_\infty^{-1}+O(1)M_\infty^{-2}+O(1)e^{-m_0M_\infty^2}.\end{aligned}$$

The proof of $\frac{e_2(\sigma)}{u_\infty}$ is similar. Finally, we combine (3.38) with (3.39) directly to obtain (3.40). This completes the proof. ∎

LEMMA 3.9. *For M_∞ sufficiently large, the following estimates hold: For any $\sigma\in[s_0,b_0]$,*

$$u_\sigma(\sigma) = \Big(\frac{b_0}{(1+b_0^2)^2}+O(1)e^{-m_0M_\infty^2}\Big)u_\infty, \tag{3.41}$$

$$v_\sigma(\sigma) = -\Big(\frac{1}{(1+b_0^2)^2}+O(1)e^{-m_0M_\infty^2}\Big)u_\infty, \tag{3.42}$$

$$u_\sigma(\sigma)+\lambda_1(\sigma)v_\sigma(\sigma) = \frac{c}{\sqrt{1+b_0^2}}+O(1)M_\infty e^{-m_0M_\infty^2}+O(1)M_\infty^{-1}e^{-m_0M_\infty^2}, \tag{3.43}$$

$$u_\sigma(\sigma)+\lambda_2(\sigma)v_\sigma(\sigma) = -\frac{c}{\sqrt{1+b_0^2}}+O(1)M_\infty e^{-m_0M_\infty^2}+O(1)M_\infty^{-1}e^{-m_0M_\infty^2}, \tag{3.44}$$

where the bound of $O(1)$ is independent of M_∞.

Proof. According to (3.3), Lemma 3.5, and Taylor's formula, we have

$$\begin{aligned}u_\sigma(\sigma) &= \frac{c^2v(\sigma)}{c^2(1+\sigma^2)-(\sigma u(\sigma)-v(\sigma))^2} = \frac{\frac{c^2b_0}{1+b_0^2}+O(1)e^{-m_0M_\infty^2}}{c^2\big(1+b_0^2\big)+O(1)e^{-m_0M_\infty^2}}u_\infty\\ &= \Big(\frac{b_0}{(1+b_0^2)^2}+O(1)e^{-m_0M_\infty^2}\Big)u_\infty.\end{aligned}$$

On the other hand, since $v_\sigma(\sigma) = -\frac{1}{\sigma}u_\sigma(\sigma)$, we finally obtain (3.42).

By Lemma 3.6 and a direct computation,

$$
\begin{aligned}
&u_\sigma(\sigma)+\lambda_1(\sigma)v_\sigma(\sigma)\\
&=\Big(\frac{b_0}{(1+b_0^2)^2}+O(1)e^{-m_0M_\infty^2}\Big)u_\infty\\
&\quad-\Big(\frac{1}{(1+b_0^2)^2}+O(1)e^{-m_0M_\infty^2}\Big)\Big(b_0-(1+b_0^2)^{\frac{3}{2}}\,M_\infty^{-1}+O(1)M_\infty^{-2}+O(1)e^{-m_0M_\infty^2}\Big)u_\infty\\
&=\frac{c}{\sqrt{1+b_0^2}}+O(1)M_\infty e^{-m_0M_\infty^2}+O(1)M_\infty^{-1}e^{-m_0M_\infty^2}.
\end{aligned}
$$

In the same way, we can prove (3.44). This completes the proof. ∎

4. Riemann Solutions for the Homogeneous System. In this section, we analyze the solutions of the Riemann problem for the homogeneous system (2.1) with piecewise constant initial data:

$$
U|_{\{x=x_0\}}=\begin{cases}U_a & \text{for } y>y_0,\\ U_b & \text{for } y<y_0,\end{cases}
\tag{4.1}
$$

where the constant states U_a and U_b denote the *above* state and *below* state with respect to line $y=y_0$, respectively, which are near the states of the background conical flow.

4.1. Riemann problem involving only weak waves. Denote by $\Gamma(b_0,u_\infty)$ the curve formed by the states on the conical flow constructed in §3; that is, $\Gamma(b_0,u_\infty)$ is the curve formed by state $(u(\sigma),v(\sigma))^\top$ that is the solution of (3.1)–(3.2). Then, on the solution curve $\Gamma(b_0,u_\infty)$ of the conical flow, we have the following properties:

LEMMA 4.1. *If $U_b\in\Gamma(b_0,u_\infty)$, then*

$$
\lim_{M_\infty\to\infty}\frac{\det\big(r_1(U_b),r_2(U_b)\big)}{M_\infty}=\frac{2c^2}{(1+b_0^2)^{\frac{5}{2}}},
\tag{4.2}
$$

$$
\lim_{M_\infty\to\infty}\frac{\det\big(r_1(U_b),r_2(U_b)\big)}{\big((u_b)_\sigma+\lambda_j(\sigma)(v_b)_\sigma\big)M_\infty}=(-1)^{j+1}\frac{2c}{(1+b_0^2)^2},\qquad j=1,2.
\tag{4.3}
$$

Proof. By Lemma 3.6, we have

$$
\begin{aligned}
\frac{\det\big(r_1(U_b),r_2(U_b)\big)}{M_\infty}&=\frac{c\,e_1(U_b)e_2(U_b)}{u_\infty}\big(\lambda_2(U_b)-\lambda_1(U_b)\big)\\
&=\frac{e_1(U_b)}{u_\infty}\frac{e_2(U_b)}{u_\infty}\Big(2c^2(1+b_0^2)^{\frac{3}{2}}+O(1)M_\infty^{-1}+O(1)M_\infty e^{-m_0M_\infty^2}\Big).
\end{aligned}
$$

Then it follows that

$$
\lim_{M_\infty\to\infty}\frac{\det\big(r_1(U_b),r_2(U_b)\big)}{M_\infty}=\frac{2c^2}{(1+b_0^2)^{\frac{5}{2}}}.
$$

Next, we turn to the proof of (4.3). By Lemma 3.9, we see that, for $j=1$,

$$
\lim_{M_\infty\to\infty}\frac{\det\big(r_1(U_b),r_2(U_b)\big)}{\big((u_b)_\sigma+\lambda_1(\sigma)(v_b)_\sigma\big)M_\infty}=\frac{2c^2}{(1+b_0^2)^{\frac{5}{2}}}\frac{\sqrt{1+b_0^2}}{c}=\frac{2c}{(1+b_0^2)^2}.
$$

The proof for $j=2$ is similar. ∎

Using the results in [29] (see also [9, 24]) and Lemma 4.1, we have the following solvability result.

PROPOSITION 4.2. *For states $\Gamma(b_0, u_\infty)$ defined above, then, for M_∞ sufficiently large, there exists a small constant $\hat{\varepsilon} > 0$ such that, for any states U_b and U_a lie in $O_{\hat{\varepsilon}}(\Gamma(b_0, u_\infty))$ with radius $\hat{\varepsilon}$ and center $\Gamma(b_0, u_\infty)$, the Riemann problem (2.1) and (4.1) admits a unique admissible solution consisting of at most two elementary waves: one for the 1-characteristic field and the other for the 2-characteristic field. Moreover, states U_b and U_a can be connected by*

$$U_a = \Phi_2(\varepsilon_2; \Phi_1(\varepsilon_1; U_b)), \tag{4.4}$$

with $\Phi_j \in C^2$, $\Phi_j|_{\varepsilon_j=0} = U_b$, and $\frac{\partial \Phi_j}{\partial \varepsilon_j}\big|_{\varepsilon_j=0} = r_j(U_b)$ for $j = 1, 2$.

Remark 4.3. For simplicity, we set

$$\Phi(\varepsilon_1, \varepsilon_2; U_b) = \Phi_2(\varepsilon_2; \Phi_1(\varepsilon_1; U_b)), \tag{4.5}$$

and denote $\{U_b, U_a\}$ as the solution of the following equation:

$$U_a = \Phi(\varepsilon_1, \varepsilon_2; U_b), \tag{4.6}$$

that is, $\{U_b, U_a\} = \{\varepsilon_1, \varepsilon_2\}$ throughout the paper.

For the statements above, the following interaction estimate was given in Glimm [17] for weak waves (also see [25, 26, 29]):

LEMMA 4.4. *Let $U_b \in \Gamma(b_0, u_\infty)$, α, β, and γ satisfy*

$$\Phi(\gamma; U_b) = \Phi(\alpha; \Phi(\beta; U_b)). \tag{4.7}$$

Then

$$\gamma = \alpha + \beta + O(1)Q^0(\alpha, \beta), \tag{4.8}$$

where

$$Q^0(\alpha, \beta) = \sum \{|\alpha_i||\beta_j| \,:\, \alpha_i \text{ and } \beta_j \text{ approach}\},$$

and $O(1)$ depends continuously on $M_\infty < \infty$.

4.2. Riemann problem involving a strong leading shock-front. Denote by $S_1(U_\infty)$ the part of the shock polar corresponding to the 1-characteristic field. Let

$$S_1^-(U_\infty) = \big\{(u, v) \in S_1(U_\infty) \,:\, c^2 \le u^2 + v^2 \le u_\infty^2,\ v < 0\big\} \qquad \text{for } U_\infty = (u_\infty, 0)^\top.$$

Following the ways in [25, 29] in a neighborhood $O_{\hat{\varepsilon}}(\Gamma(b_0, u_\infty))$ of $\Gamma(b_0, u_\infty)$, we can parameterize the shock polar $S_1^-(U_\infty) \cap O_{\hat{\varepsilon}}(\Gamma(b_0, u_\infty))$ for the homogeneous system (2.1) through U_∞ by a C^2–function $\Theta : s \mapsto \Theta(s, U_\infty)$; that is, $\Theta(s, U_\infty)$ is the state that can be connected to U_∞ by a shock with slope s and left-state U_∞. In the following, we write $\Theta(s, U_\infty)$ as $\Theta(s)$ in the sequel for simplification, and denote by $\tilde{u}(s)$ and $\tilde{v}(s)$ as the components of $\Theta(s)$, *i.e.*, $\Theta(s) = (\tilde{u}(s), \tilde{v}(s))^\top$. Moreover, on the shock polar, we have

LEMMA 4.5. *For M_∞ sufficiently large, the following expansions hold*:

$$\frac{\tilde{u}(s_0)}{u_\infty} = \frac{1}{1+b_0^2} + O(1)e^{-m_0 M_\infty^2}, \tag{4.9}$$

$$\frac{\tilde{v}(s_0)}{u_\infty} = \frac{b_0}{1+b_0^2} + O(1)e^{-m_0 M_\infty^2}, \tag{4.10}$$

$$\frac{\tilde{u}_s(s_0)}{u_\infty} = -\frac{2b_0}{(1+b_0^2)^2} + O(1)M_\infty^2 e^{-m_0 M_\infty^2} + O(1)e^{-m_0 M_\infty^2}, \tag{4.11}$$

$$\frac{\tilde{v}_s(s_0)}{u_\infty} = \frac{1-b_0^2}{(1+b_0^2)^2} + O(1)M_\infty^2 e^{-m_0 M_\infty^2} + O(1)e^{-m_0 M_\infty^2}, \tag{4.12}$$

and, for $j = 1, 2$,

$$\begin{aligned}\frac{\tilde{u}_s(s_0) + \lambda_j(s_0)\tilde{v}_s(s_0)}{u_\infty} &= -\frac{b_0}{1+b_0^2} + (-1)^j(1-b_0^2)(1+b_0^2)^{-\frac{1}{2}}\, M_\infty^{-1} + O(1)M_\infty^{-2} \\ &\quad + O(1)M_\infty^2 e^{-m_0 M_\infty^2} + O(1)e^{-m_0 M_\infty^2},\end{aligned} \tag{4.13}$$

$$\begin{aligned}\frac{\tilde{u}_s(s_0) + \lambda_1(s_0)\tilde{v}_s(s_0)}{\tilde{u}_s(s_0) + \lambda_2(s_0)\tilde{v}_s(s_0)} &= 1 + 2b_0^{-1}(1-b_0^2)\sqrt{1+b_0^2}\, M_\infty^{-1} + O(1)M_\infty^{-2} \\ &\quad + O(1)M_\infty^2 e^{-m_0 M_\infty^2} + O(1)e^{-m_0 M_\infty^2},\end{aligned} \tag{4.14}$$

where $\tilde{u}_s(s_0) = \frac{\partial \tilde{u}}{\partial s}(s_0), \tilde{v}_s(s_0) = \frac{\partial \tilde{v}}{\partial s}(s_0)$, and the bound of $O(1)$ is independent of M_∞.

Proof. The first two expansions are directly from Lemma 3.5. To obtain the other expansions, we first see that, on the shock polar, the Rankine-Hugoniot conditions hold:

$$\tilde{\rho}(s)\big(\tilde{u}(s)s - v(s)\big) = u_\infty s, \tag{4.15}$$

$$\tilde{u}(s) + \tilde{v}(s)s = u_\infty, \tag{4.16}$$

and the Bernoulli law

$$\frac{\tilde{u}^2(s) + \tilde{v}^2(s)}{2} + c^2 \ln \tilde{\rho}(s) = \frac{u_\infty^2}{2}. \tag{4.17}$$

We take derivative to (4.15)–(4.17) with respect to s and then let $s = s_0$ to obtain

$$\begin{aligned} A_{11}(s_0)\frac{\tilde{u}_s(s_0)}{u_\infty} + A_{12}(s_0)\frac{\tilde{v}_s(s_0)}{u_\infty} &= B_1(s_0), \\ A_{21}(s_0)\frac{\tilde{u}_s(s_0)}{u_\infty} + A_{22}(s_0)\frac{\tilde{v}_s(s_0)}{u_\infty} &= B_2(s_0), \end{aligned}$$

where

$$A_{11}(s_0) = \frac{s_0(\tilde{u}^2(s_0) - c^2) - \tilde{u}(s_0)\tilde{v}(s_0)}{c^2}, \quad A_{12}(s_0) = \frac{c^2 - \tilde{v}^2(s_0) + s_0\tilde{u}(s_0)\tilde{v}(s_0)}{c^2},$$

$$A_{21}(s_0) = -1, \quad A_{22}(s_0) = -s_0, \quad B_1(s_0) = \frac{\tilde{\rho}(s_0)\tilde{u}(s_0) + u_\infty}{\tilde{\rho}(s_0)u_\infty}, \quad B_2(s_0) = \frac{\tilde{v}(s_0)}{u_\infty}.$$

For M_∞ sufficiently large, it follows from (4.9)–(4.10) that

$$
\begin{aligned}
&A_{11}(s_0) = -b_0 + O(1)M_\infty^2 e^{-m_0 M_\infty^2} + O(1)e^{-m_0 M_\infty^2},\\
&A_{12}(s_0) = 1 + O(1)M_\infty^2 e^{-m_0 M_\infty^2}, \qquad A_{22}(s_0) = -b_0 + O(1)e^{-m_0 M_\infty^2},\\
&B_1(s_0) = \frac{1}{1+b_0^2} + O(1)e^{-m_0 M_\infty^2}, \qquad B_2(s_0) = \frac{b_0}{1+b_0^2} + O(1)e^{-m_0 M_\infty^2}.
\end{aligned}
$$

Thus, by Cramer's rule, we have

$$
\begin{aligned}
\frac{\tilde{u}_s(s_0)}{u_\infty} &= \frac{A_{22}(s_0)B_1(s_0) - A_{12}(s_0)B_2(s_0)}{A_{11}(s_0)A_{22}(s_0) - A_{12}(s_0)A_{21}(s_0)}\\
&= -\frac{2b_0}{(1+b_0^2)^2} + O(1)M_\infty^2 e^{-m_0 M_\infty^2} + O(1)e^{-m_0 M_\infty^2},\\
\frac{\tilde{v}_s(s_0)}{u_\infty} &= \frac{A_{11}(s_0)B_2(s_0) - A_{21}(s_0)B_1(s_0)}{A_{11}(s_0)A_{22}(s_0) - A_{12}(s_0)A_{21}(s_0)}\\
&= \frac{1-b_0^2}{(1+b_0^2)^2} + O(1)M_\infty^2 e^{-m_0 M_\infty^2} + O(1)e^{-m_0 M_\infty^2}.
\end{aligned}
$$

Next, for $j = 1$, we use Lemma 3.6 and (4.11)–(4.12) to obtain

$$
\begin{aligned}
\frac{\tilde{u}_s(s_0) + \lambda_j(s_0)\tilde{v}_s(s_0)}{u_\infty} &= -\frac{2b_0}{(1+b_0^2)^2} + O(1)M_\infty^2 e^{-m_0 M_\infty^2} + O(1)e^{-m_0 M_\infty^2}\\
&\quad + \Big(b_0 - (1+b_0^2)^{\frac{3}{2}} M_\infty^{-1} + O(1)M_\infty^{-2} + O(1)e^{-m_0 M_\infty^2}\Big)\\
&\qquad \times \Big(\frac{1-b_0^2}{(1+b_0^2)^2} + O(1)M_\infty^2 e^{-m_0 M_\infty^2} + O(1)e^{-m_0 M_\infty^2}\Big)\\
&= -\frac{b_0}{1+b_0^2} - (1-b_0^2)(1+b_0^2)^{-\frac{1}{2}} M_\infty^{-1} + O(1)M_\infty^{-2}\\
&\quad + O(1)M_\infty^2 e^{-m_0 M_\infty^2} + O(1)e^{-m_0 M_\infty^2}.
\end{aligned}
$$

The case, $j = 2$, can be handled similarly. Finally, by the Taylor formula, we have

$$
\begin{aligned}
&\frac{\tilde{u}_s(s_0) + \lambda_1(s_0)\tilde{v}_s(s_0)}{\tilde{u}_s(s_0) + \lambda_2(s_0)\tilde{v}_s(s_0)}\\
&= \frac{\frac{b_0}{1+b_0^2} + (1-b_0^2)(1+b_0^2)^{-\frac{1}{2}} M_\infty^{-1} + O(1)M_\infty^{-2} + O(1)M_\infty^2 e^{-m_0 M_\infty^2} + O(1)e^{-m_0 M_\infty^2}}{\frac{b_0}{1+b_0^2} - (1-b_0^2)(1+b_0^2)^{-\frac{1}{2}} M_\infty^{-1} + O(1)M_\infty^{-2} + O(1)M_\infty^2 e^{-m_0 M_\infty^2} + O(1)e^{-m_0 M_\infty^2}}\\
&= 1 + 2b_0^{-1}(1-b_0^2)\sqrt{1+b_0^2}\, M_\infty^{-1} + O(1)M_\infty^{-2} + O(1)M_\infty^2 e^{-m_0 M_\infty^2} + O(1)e^{-m_0 M_\infty^2}.
\end{aligned}
$$

This completes the proof. ∎

According to Lemma 4.5, we can obtain the solvability of the above Riemann problem near the strong shock as below:

PROPOSITION 4.6. *For M_∞ sufficiently large, there exists a constant $\delta_0 > 0$ such that, for states $U_b = U_\infty$ and $U_a \in O_{\hat{\varepsilon}}(\Gamma(b_0, u_\infty)) \cap O_{\delta_0}(\Theta(s_0))$, the Riemann problem (2.1) and (4.1) admits a unique admissible solution that contains a strong 1-shock and a 2-weak wave of the 2-characteristic field – either a 2-shock or 2-rarefaction wave.*

5. Construction of Approximate Solutions. In this section, we construct global approximate solutions of the initial-boundary value problem (1.1)–(1.7) under the assumptions of Theorem 1.2. We develop a modified Glimm scheme with the Riemann solutions of the homogeneous system (2.1) and the local self-similar solutions of problem (3.1)–(3.2) as building blocks in order to incorporate the geometric source term.

To do this, denote Δx and Δy as mesh lengths in x and y respectively, and $\Delta\sigma$ as a uniform grid size for the self-similar variable σ. The initial numerical grid size Δx and $\Delta\sigma$ are suitably chosen so that the usual Courant-Friedrichs-Lewy condition holds:

$$\frac{\Delta y}{\Delta x} > 2\max_{i=1,2}\big\{\sup_{U}|\lambda_i(U)|\big\}.$$

We also choose a set of points $\{A_k\}_{k=0}$ with $A_k = (x_k, b_k)$, where $x_k = x_0 + k\Delta x$ and $b_k = b(x_k)$ for $k = 0, 1, \cdots$.

Define

$$b_\Delta(x) = b_k + \frac{b_{k+1} - b_k}{\Delta x}(x - x_k) \qquad \text{for } x \in [x_k, x_{k+1}) \text{ and } k \geq 0.$$

Let

$$\Omega_{\Delta x,k} = \big\{(x,y) : x_k \leq x < x_{k+1},\ y < b_\Delta(x)\big\},\ \ \Omega_{\Delta x} = \big\{(x,y) : x > 0,\ y < b_\Delta(x)\big\},$$

$$\Gamma_{\Delta x,k} = \big\{(x,y) : x_k \leq x < x_{k+1},\ y = b_\Delta(x)\big\},\ \ \Gamma_{\Delta x} = \big\{(x,y) : x > 0,\ y = b_\Delta(x)\big\}.$$

Denote

$$\theta_0 = \arctan b_0, \qquad \theta_k = \arctan\Big(\frac{b_k - b_{k-1}}{\Delta x}\Big) \ \text{ for } k > 0,$$
$$\omega_0 = \arctan\Big(\frac{b(x_0) - b(0)}{x_0}\Big), \qquad \omega_k = \theta_{k+1} - \theta_k \ \text{ for } k \geq 0,$$

so that ω_k represents the change of angle at the turning point A_k for each $k \geq 0$.

From hypothesis ($\mathbf{H_1}$), when $x > x_0$, the cone boundary is approximated by a set of line segments $\Gamma_{\Delta x,k}$ with $\Gamma_{\Delta x,k} = A_kA_{k+1}$ for $k \geq 0$, so that the slope of $\Gamma_{\Delta x,k}$ is negative and uniformly bounded. Then we can extend $\Gamma_{\Delta x,k}$ so that the extension of $\Gamma_{\Delta x,k}$ and the x–axis intersect at point $(X_k^*, 0)$ with

$$X_k^* = x_{k-1} - \frac{b_{k-1}}{b_k - b_{k-1}}\Delta x; \tag{5.1}$$

and point $(X_k^*, 0)$ is called the center of the self-similar solution for each $k \geq 0$. Moreover, by a direct computation, we have

LEMMA 5.1. *For $k > 0$,*

$$X_k^* - X_{k-1}^* = O(1)b_{k-1}(\tan\theta_k - \tan\theta_{k-1}), \tag{5.2}$$

where $O(1)$ depends only on b_0, independent of k.

Proof. By (5.1), we have

$$\begin{aligned} X_k^* - X_{k-1}^* &= x_{k-1} - \frac{b_{k-1}}{b_k - b_{k-1}}\Delta x - \Big(x_{k-2} - \frac{b_{k-2}}{b_{k-1} - b_{k-2}}\Delta x\Big) \\ &= \Big(1 - \frac{b_{k-1}}{b_k - b_{k-1}} + \frac{b_{k-2}}{b_{k-1} - b_{k-2}}\Big)\Delta x \\ &= \frac{b_{k-1}}{\tan\theta_k \tan\theta_{k-1}}(\tan\theta_k - \tan\theta_{k-1}), \end{aligned}$$

which leads to the desire result by assumption $(\mathbf{H}_1)$. ∎

We now describe the construction of the difference scheme and corresponding approximate solutions. In region $\{(x,y) : 0 < x \le x_0, y < b_0 x\}$, the approximate solution is defined as the unperturbed conical flow with center at $(0,0)$. For $x = x_0$, the grid points are the intersection points of $x = x_0$ with the self-similar rays centered at $(0,0)$:

$$y = (\tan\omega_0 + h\Delta\sigma)x \qquad \text{for } h = 0, -1, -2, \cdots.$$

Choose an equi-distributed sequence $\vartheta = (\vartheta_0, \cdots, \vartheta_k, \cdots) \in \Pi_{k=0}^{\infty}(-1,1)$. Suppose that the approximate solution $U_{\Delta x,\vartheta}(x,y)$ has been defined for $x < x_k$, and the grid points have been defined for $x \le x_k$ for $k \ge 1$. The approximate solution $U_{\Delta x,\vartheta}(x_l, y)$ is a piecewise smooth solution of problem (3.1)–(3.2) on each vertical grid line $x = x_l+$ for $l < k$. That is, at any continuous point (x,y) of this approximate solution, it has the form:

$$U_{\Delta x,\vartheta}(x,y) = U_{\text{self}}(\sigma(x,y)),$$

where $\sigma(x,y) = \frac{y}{x - X^*}$, $U_{\text{self}}(\sigma)$ is a self-similar solution of system (3.1)–(3.2), and $X^* = X^*(x,y)$ (called the center of U_{self}) is a piecewise constant and right-continuous function. As part of the induction hypothesis, we also assume that center X^* of the constructed self-similar solution has been specified on $\{x = x_l,\ y_{h-1}(l) < y < y_h(l)\}$ for $l < k,\ h = 0, -1, \cdots$, and $X^* \in \{X^*\}_{j\ge 0}$ for $x < x_k$, where $y = y_h(l)$ is the grid points on $x = x_l$ and $y_0(l) = b(x_l)$.

Then we define the approximate solution $U_{\Delta x,\vartheta}(x,y)$ and the numerical grids inductively for regions $\Omega_{\Delta x,k}$ for $k \ge 1$. The construction of the approximate solution on $\Omega_{\Delta x}$ between $x_k \le x < x_{k+1}$ is based on the following three cases:

5.1. Case 1: Away from the cone boundary in region $\{x_k \le x < x_{k+1}\} \cap \Omega_{\Delta x}$. We construct the approximate solution $U_{\Delta x,\vartheta}(x,y)$ in the following four steps:

(i) Define the approximate solution on any interval $y_h(k) < y < y_{h+1}(k), h \le -1$, on line $x = x_k$. Let $U(x_k, y)$ be the solution of system (3.1)–(3.2) with the following initial data given at the mesh point:

$$U(x_k, a_{k,h}) = U_{\Delta x,\vartheta}(x_k - 0, a_{k,h}), \tag{5.3}$$

where $a_{k,h}$ is a random choice point and can be represented as $y_h(k) + \vartheta_k(y_{h+1}(k) - y_h(k))$. This is a Cauchy problem (3.1)–(3.2) of the ordinary differential system, whose solution is of self-similar with variable σ that is defined below. It should be noted that the initial value above does not uniquely determine the non-autonomous system (3.3), and the center of the self-similar solution need to be specified. We specify center

$X^*(x_k, y)$ to be the center of the self-similar solution $U_{\Delta x,\vartheta}(x_k - 0, a_{k,h})$ through the random choice method. That is,

$$X^*(x_k, y) = X^*(x_k-, a_{k,h}) \qquad \text{for } y_h(k) < y < y_{h+1}(k).$$

In other words, when the center on line $x = x_k - 0$ is defined, the center for $\{x = x_k + 0,\ y_h(k) < x < y_{h+1}(k)\}$ is the same as the center for $(x_k - 0, a_{k,h})$. Then this yields the self-similar variable:

$$\sigma = \frac{y}{x_k - X^*(x_k, y)}.$$

(ii) The approximate solution on line $x = x_k$ defined above may have discontinuities on the grid points $(x_k, y_h(k)), h = -1, -2, \cdots$. Therefore, we construct $U_{\Delta x,\vartheta}(x, y)$ in $\{(x, y) : x_k < x < x_{k+1},\ y_h(k) < y < y_{h+1}(k)\}$ by solving a series of Riemann problems of system (2.1) with the initial data:

$$U_{\Delta x,\vartheta}(x, y) = \begin{cases} U_{\Delta x,\vartheta}(y_h(k) + 0, x_k + 0) & \text{for } y > y_h(k), \\ U_{\Delta x,\vartheta}(y_h(k) - 0, x_k + 0) & \text{for } y < y_h(k). \end{cases} \tag{5.4}$$

That is, if $U_{\Delta x,\vartheta}(y_h(k) \pm 0, x_k + 0) \in O_{\tilde{\varepsilon}}\big(\Gamma(b_0, u_\infty)\big) \cap O_{\delta_0}(\Theta(s_0))$, then it follows from Proposition 4.1 that this Riemann problem is solvable, and the solution is a function of $\xi = \frac{y - y_h(k)}{x - x_k}$ and consists of shocks and/or rarefaction waves.

(iii) To include the information of the geometric lower order term, we make a so-called *self-similar modification* for the approximate solution constructed above.

Let

$$\sigma = \sigma(x, y) = \frac{y}{x - X^*(x, y)}.$$

From the above steps, $\sigma(x, y)$ is well defined and satisfies

$$\lambda_1(U_{\Delta x,\vartheta}(x, y)) < \sigma(x, y) < \lambda_2(U_{\Delta x,\vartheta}(x, y)).$$

Denote

$$\sigma_{h-\frac{1}{2}}(k) = \sigma(x_k-, \frac{y_{h-1} + y_h}{2}).$$

Then, along ray $\frac{y - y_h(k)}{x - x_k} = \xi$ for each ξ, the approximate solution $U_{\Delta x,\vartheta}(x, y)$ in $\{x_k < x < x_{k+1}, \sigma_{h-\frac{1}{2}}(k) < \sigma(x, y) < \sigma_{h+\frac{1}{2}}(k)\}$ is defined as the solution of equation (3.3) with the initial data $U(\xi)$ at $x = x_k + 0$, where

$$\sigma = \frac{y}{x - X^*_{k,h}}, \quad X^*_{k,h} = X(x_k-, a_{k,h}) \qquad \text{for } \xi > \xi_{k,h},$$

$$\sigma = \frac{y}{x - X^*_{k,h-1}}, \quad X^*_{k,h-1} = X(x_k-, a_{k,h-1}) \qquad \text{for } \xi < \xi_{k,h},$$

and $U(\xi)$ is the solution of the Riemann problem given above. For this, the center keeps invariant along the rays.

(iv) Finally, as in [22], the grid lines between $x = x_k$ and $x = x_{k+1}$ are defined by the rays going through every grid point on $x = x_k$, and the numerical grid points on $x = x_{k+1}$ are defined to be the interaction points between the corresponding grid lines and $x = x_{k+1}$. The new centers on $x = x_{k+1}$ inherit those centers on $x = x_k+$ through the random choice. Then we obtain the approximate solution in region $\{x_k \le x < x_{k+1}\} \cap \Omega_{\Delta x}$ and extend it to the whole domain $\Omega_{\Delta x}$ by induction.

5.2. Case 2: On the cone boundary $\{x_k \le x < x_{k+1}\} \cap \Gamma_{\Delta x}$. In general, a 1-wave is produced and emerges into the domain owing to the turning angle of the cone boundary. It can be a shock or rarefaction wave depending on the change of the boundary angle toward (or away from) the flow. Meanwhile, the 2-wave issuing from $(x_k, y_{-1}(k))$ is reflected on the boundary, and a 1-wave is formed. We define the approximate solution in the following three steps:

(i) As before, we first solve problem (3.1)–(3.2) on $\{x = x_k,\ y_{-1}(k) < y < y_0(k)\}$, where the center is chosen as the same as the center of the initial data.

(ii) Next, we solve the initial-boundary problem of system (2.1) with initial data:

$$U(x_k, y) = U_{\Delta x, \vartheta}(x_k - 0, y_0(k)-) \qquad \text{for } y_{-1}(k) < y < y_0(k),$$

and with the boundary condition on $\Gamma_{\Delta x, k}$:

$$v = \sigma_0(k) u,$$

where

$$\sigma_0(k) = \frac{y_0(k+1) - y_0(k)}{x_{k+1} - x_k}.$$

The solution of this problem contains only a 1-wave. Between the lower edge of the 1-wave and the cone boundary, the center is chosen as the intersection of the ray through $(x_k, y_0(k))$ with slope $\sigma_0(k)$ and the x–axis, *i.e.*, $(x_k - \frac{y_0(k)}{\sigma_0(k)}, 0)$. We point out that the center changes of the self-similar solutions in the whole domain between the cone boundary and the leading shock-front are due to the changes of the cone boundary slopes. As to the centers below the lower edge of the 1-wave, it has been defined for Case 1 in §5.1.

(iii) We also make a *self-similar modification* for this solution as in **Case 1**. Then the approximate solution is extended to

$$\Big\{(x, y) : x_k \le x < x_{k+1},\ y_0(k) + \frac{1}{2}(y_0 - y_{-1}(k)) < y < y_0(k) + \sigma_0(k)(x - x_k)\Big\}$$

as before with center $x_k - \frac{y_0(k)}{\sigma_0(k)}$.

5.3. Case 3: Near the leading conical shock-front next to the uniform upstream flow traced continuously. Suppose that the approximate solution has been constructed for $x < x_k$. Let $(x, y_{\rho s}(x))$ be the locus of the front of the strong leading 1-shock. Suppose that $y_{h_s - 1}(k) < y_s(x) < y_{h_s + 1}(k)$. As in [22, 25], the interval $y_{h_s - 1}(k) < y_s(x) < y_{h_s + 1}(k)$ called the front region at $x = x_k$. Inside the front region, we first solve the self-similar solution of system (3.3) with the initial data:

$$U(x_k, a_{k, h_s}) = U_{\Delta x, \vartheta}(x_k - 0, a_{k, h_s})$$

and self-similar variable:

$$\sigma = \frac{y}{x_k - X^*(x_k, a_{k, h_s})}.$$

The solution is denoted as $U_{\text{self}}(x_k, y)$ that satisfies

$$\lambda_1(U_{\text{self}}(x_k, y)) < \sigma(x_k, y) < \lambda_2(U_{\text{self}}(x_k, y)).$$

Next, we solve the Riemann problem of system (2.2) with the initial data:

$$U|_{\{x=x_k\}} = \begin{cases} U_{\rm self}(x_k, y) & \text{for } y_{\rm s}(k) < y < y_{h_{\rm s}+1}(k), \\ U_\infty & \text{for } y < y_{\rm s}(k). \end{cases}$$

The solution, $U(x,y)$, contains a weak 2-wave and a strong 1-shock denoted by $y = \chi_{\Delta x,\vartheta}(x)$ with speed s_{k+1}. Solve equation (3.3) again on interval $y_{\rm s}(k) < y < y_{h_{\rm s}+1}(k)$ with initial value $U(x_k, y_{\rm s}(k)+) = U_+$ and self-similar variable $\sigma = \frac{y}{x_k - X^*(x_k, a_{k,h_{\rm s}})}$. Denote its solution by $U_-(\sigma)$. Now we can define the approximate solution in the front region:

$$U_{\Delta x,\vartheta}(x,y) = \begin{cases} U_-(\sigma) & \text{for } y_{\rm s}(k) < y < y_{h_{\rm s}+1}(k), \\ U_\infty & \text{for } y < y_{\rm s}(k). \end{cases}$$

The discontinuities at point $(x_k, y_{h_{\rm s}+1})$ are resolved in the same way as in **Case 1**. Moreover, we must also specify the center of the self-similar variable near the leading shock-front as:

$$X^*(x,y) = X^*_{k,h_{\rm s}} \qquad \text{for } x_k < x < x_{k+1} \text{ and } s_{k+1} < \tfrac{y - y_{h_{\rm s}+1}(k)}{x - x_k} < \xi_{k,h_{\rm s}+1}.$$

In this way, we complete the construction of the difference scheme and corresponding approximate solutions $U_{\Delta x,\vartheta}(x,y)$ globally in $\Omega_{\Delta x} \cup \Gamma_{\Delta x}$.

6. Local Interaction Estimates. In this section, we establish some uniform estimates of the approximate solutions constructed in §5. For these, the following formulas are used:

(i) If $f \in C^1(\mathbb{R})$, then

$$f(x) - f(0) = x \int_0^1 f_x(\mu x)\, {\rm d}\mu \qquad \text{for any } x \in \mathbb{R}. \tag{6.1}$$

(ii) If $f \in C^2(\mathbb{R}^2)$, then, for any $(x,y) \in \mathbb{R}^2$,

$$f(x,y) - f(x,0) - f(0,y) + f(0,0) = xy \int_0^1 \int_0^1 f_{xy}(\mu x, \tau y)\, {\rm d}\mu {\rm d}\tau. \tag{6.2}$$

LEMMA 6.1. *Let $\sigma_k = \frac{y_k}{x_k - X}$ and $\bar\sigma_k = \frac{y_k}{x_k - \bar X}$ for $k = 1, 2$, with $X, \bar X > 0$ as their centers. Then*

$$\Delta\bar\sigma = \Delta\sigma + O(1)|X - \bar X||\Delta\sigma| + O(1)|X - \bar X||x_2^{-1} - x_1^{-1}|, \tag{6.3}$$

where $\Delta\sigma = \sigma_2 - \sigma_1$, $\Delta\bar\sigma = \bar\sigma_2 - \bar\sigma_1$, and $O(1)$ is independent of X, $\bar X$, and $\Delta\sigma$.

Proof. Since

$$\bar\sigma_j - \sigma_j = \frac{y_j}{x_j - \bar X} - \frac{y_j}{x_j - X} = \frac{y_j}{(x_j - X)(x_j - \bar X)}(X - \bar X) \qquad \text{for } j = 1, 2,$$

then

$$\begin{aligned}\Delta\bar{\sigma}-\Delta\sigma&=\Big(\frac{y_2}{(x_2-\bar{X})(x_2-X)}-\frac{y_1}{(x_1-\bar{X})(x_1-X)}\Big)(\bar{X}-X)\\&=\frac{1}{x_2-\bar{X}}\Big(\frac{y_2}{x_2-X}-\frac{y_1}{x_1-X}\Big)(\bar{X}-X)\\&\quad+\frac{y_1}{x_1-X}\frac{x_1x_2}{(x_1-\bar{X})(x_2-\bar{X})}\big(x_2^{-1}-x_1^{-1}\big)(\bar{X}-X)\\&=O(1)|X-\bar{X}||\Delta\sigma|+O(1)|X-\bar{X}||x_2^{-1}-x_1^{-1}|,\end{aligned}$$

where $O(1)$ is independent of X, $\bar{X}$, and $\Delta\sigma$. ∎

From now on, we use Greek letters α, β, γ, and δ to represent the elementary waves in the approximate solution, and α_i, β_i, γ_i, and δ_i, $i=1,2$, denote the corresponding i-th components of the respective waves. To avoid the confusion, when $U(x,y)=(u,v)^\top(x,y)$ is the solution of problem (3.1)–(3.2), we often use $\varpi(\sigma;O)$ to stand for the states, where $\sigma=\frac{y}{x-X_k^*}$ is the self-similar variable with $O=(X_k^*,0)$ as the corresponding center. In addition, we use $\Psi=\Psi(\sigma-\sigma_0,\sigma_0;\varpi(\sigma_0;O))$ to be the solution of system (3.3) with initial data:

$$\Psi|_{\sigma=\sigma_0}=\varpi(\sigma_0;O),$$

where $\varpi(\sigma_0;O)\in O_{\hat{\varepsilon}}(\Gamma(b_0,u_\infty))$.

As in [9, 24, 25, 29], a curve I is called a mesh curve if I is a space-like curve that consists of the line segments joining the random points one by one in turn. Then I divides region $\Omega_{\Delta x}$ into two parts: I^- and I^+, where I^- denotes the part containing line $x=x_0$. For any two mesh curves I and J, we use $J>I$ to represent that every mesh point of curve J is either on I or contained in I^+. In particular, we call J an immediate successor to I, provided that $J>I$, and every mesh point of J except the one is on I.

Let

$$\Omega^+_{\Delta x,j}=\Omega_{\Delta x,j}\cap\{y>\chi_{\Delta x,\vartheta}(x)\},\qquad \Omega^-_{\Delta x,j}=\Omega_{\Delta x,j}\cap\{y<\chi_{\Delta x,\vartheta}(x)\},$$

where curve $y=\chi_{\Delta x,\vartheta}(x)$ is the approximate strong leading shock-front with speed $s_{\Delta x,\vartheta}(x)$.

We make the following inductive hypotheses:

$(\mathbf{P}1)_{(k)}$ The approximate solution $U_{\Delta x,\vartheta}(x,y)$ has been defined in

$$\{0\le x\le k\Delta x\}\cap\Omega_{\Delta x};$$

$(\mathbf{P}2)_{(k)}$ For any $(x,y)\in\Omega^+_{\Delta x,j}$,

$$U_{\Delta x,\vartheta}\in O_{\hat{\varepsilon}}(\Gamma(b_0,u_\infty))\cap O_{\delta_0}(\Theta(s_0)),$$

and, for any $(x,y)\in\Omega^-_{\Delta x,j}$ with $0\le j\le k$,

$$U_{\Delta x,\vartheta}=U_\infty;$$

$(\mathbf{P}3)_{(k)}$ For any weak wave α,

$$\lambda_1(U_{\Delta x,\vartheta}(x_\alpha-,\cdot))<\sigma(x_\alpha-,\cdot)<\lambda_2(U_{\Delta x,\vartheta}(x_\alpha-,\cdot)),$$

where (x_α, y_α) denotes the point with $x_\alpha \in \{x_j : 0 \le j \le k\}$ from where the weak wave α issues, while $\sigma(x_\alpha, y_\alpha)$ represents the corresponding self-similar variable, and $U_{\Delta x,\vartheta}(x, y)$ stands for its approximate solution.

Then we prove in the following subsections that, under suitable conditions, $U_{\Delta x,\vartheta}$ can be defined in $\{0 \le x \le (k+1)\Delta x\} \cap \Omega_{\Delta x}$ and satisfies $(\mathbf{P}1)_{(k+1)}$–$(\mathbf{P}3)_{(k+1)}$. As in [17] (also see [9, 16, 24]), we carry out this step by considering any pairs of the mesh curves I and J with J as an immediate successor to I, where I and J are in $\{(k-1)\Delta x \le x \le (k+1)\Delta x\} \cap \Omega_{\Delta x}$.

Now let Λ be the diamond between I and J. Suppose that

$$U_{\Delta x,\vartheta}(x, y) \in O_{\hat{\varepsilon}}(\Gamma(b_0, u_\infty)) \cap O_{\delta_0}(\Theta(s_0)) \quad \text{for any } (x, y) \in I \cap (\Omega^+_{\Delta x,k} \cup \Omega_{\Delta x,k+1}).$$

and

$$\lambda_1(U_{\Delta x,\vartheta}(x_\alpha-, \cdot)) < \sigma(x_\alpha-, \cdot) < \lambda_2(U_{\Delta x,\vartheta}(x_\alpha-, \cdot)) \text{ for any weak wave } \alpha \text{ crossing } I.$$

6.1. Λ is between $y = b_\Delta(x)$ and $y = \chi_{\Delta x,\vartheta}(x)$. In this section, we consider the interactions involving only weak waves. By the construction of the approximate solution, the waves entering Λ are denoted by $\alpha = (\alpha_1, 0)$ and $\beta = (\beta_1, \beta_2)$ that issue from $(x_{k-1}, y_h(k-1))$ and $(x_{k-1}, y_{h-1}(k-1))$, respectively. Let $\delta = (\delta_1, \delta_2)$ be the set of waves issuing from $(x_k, y_{h-1}(k))$ (see Fig. 6.1).

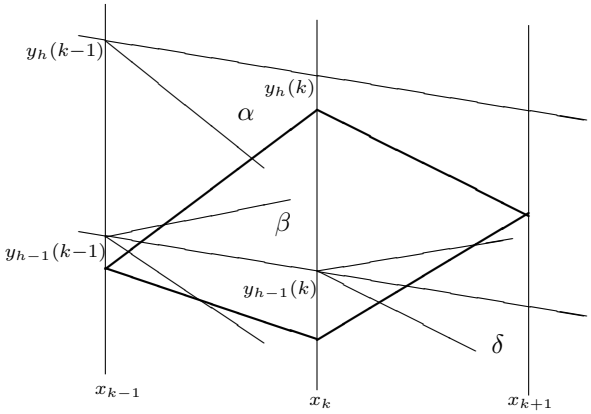

Fig. 6.1. *Interactions involving only weak waves*

We now consider the case:

$$\varpi(\sigma_1; O_1) = \Phi(\alpha; \varpi(\bar{\sigma}_1; O_2)), \tag{6.4}$$

$$\varpi(\bar{\sigma}_2; O_2) = \Phi(\beta; \varpi(\tilde{\sigma}_3; O_3)), \tag{6.5}$$

$$\varpi(\sigma_2; O_1) = \Phi(\delta; \varpi(\tilde{\sigma}_3; O_3)), \tag{6.6}$$

where $O_1 = (X, 0)$, $O_2 = (\bar{X}, 0)$, and $O_3 = (\tilde{X}, 0)$.

For notational convenience, we denote $\tilde{x}_0 := |\bar{X} - X|, \tilde{x}_1 := |\tilde{X} - \bar{X}|$, and $U_b := \varpi(\tilde{\sigma}_3; O_3)$ from now on.

LEMMA 6.2. *For the waves described above,*

$$\delta = \alpha + \beta + O(1)Q(\Lambda), \tag{6.7}$$

where $Q(\Lambda)=Q^0(\Lambda)+Q^1(\Lambda)+Q^c(\Lambda)$ *with*

$$\begin{aligned}
Q^0(\Lambda)&=\sum\big\{|\alpha_i||\beta_j|\,:\,\alpha_i \text{ and } \beta_j \text{ approach}\big\},\\
Q^1(\Lambda)&=|\alpha||\Delta\sigma|,\\
Q^c(\Lambda)&=\big(|\Delta\sigma|+|x_k^{-1}-x_{k-1}^{-1}|\big)\tilde{x}_0,
\end{aligned}$$

and $\Delta\sigma=\sigma_1-\sigma_2$, *and* $O(1)$ *depends continuously on* M_∞ *but independent of* $\alpha,\beta,\Delta\sigma$, *and* x_0.

Proof. We combine (6.4)–(6.6) to obtain

$$\Psi(\Delta\sigma,\sigma_2;\Phi(\delta;U_b))=\Phi(\alpha;\Psi(\Delta\bar{\sigma},\bar{\sigma}_2;\Phi(\beta;U_b))). \tag{6.8}$$

Lemma 3.9 yields

$$\begin{aligned}
&\lim_{M_\infty\to\infty}\det\Big(\frac{\partial\Phi\big(\delta;U_b\big)}{\partial(\delta_1,\delta_2)}\Big|_{\{\alpha=\beta=\Delta\sigma=\Delta\bar{\sigma}=x_0=0\}}\Big)\frac{1}{M_\infty}\\
&=\lim_{M_\infty\to\infty}\frac{\det(r_1(U_b),r_2(U_b))}{M_\infty}=\frac{2c^2}{(1+b_0^2)^{\frac{5}{2}}}\neq 0.
\end{aligned}$$

Then, by the implicit function theorem, system (4.7) has a unique C^2–solution:

$$\delta=\delta(\alpha,\beta,\Delta\sigma,\Delta\bar{\sigma},x_0;U_b)$$

in a neighborhood of $(\alpha,\beta,\Delta\sigma,\Delta\bar{\sigma},x_0;U_b)=(0,0,0,0,0;U_\infty)$.

Let $\delta'=\delta(\alpha,\beta,\Delta\sigma,\Delta\sigma,x_0;U_b)$. By (6.1), we have

$$\delta=\delta'+K'|\Delta\bar{\sigma}-\Delta\sigma|, \tag{6.9}$$

where $K'=\displaystyle\int\frac{\partial\delta}{\partial(\Delta\bar{\sigma})}\,\mathrm{d}\xi$, and δ' solves the equation:

$$\Psi(\Delta\sigma,\sigma_2;\Phi(\delta';U_b))=\Phi(\alpha;\Psi(\Delta\sigma,\bar{\sigma}_2;\Phi(\beta;U_b))).$$

Moreover, by Lemma 3.9 again,

$$\begin{aligned}
&\lim_{M_\infty\to\infty}\Big(M_\infty\,\frac{\partial\delta}{\partial(\Delta\bar{\sigma})}\Big|_{\{\alpha=\beta=\Delta\sigma=\Delta\bar{\sigma}=x_0=0\}}\Big)\\
&=\lim_{M_\infty\to\infty}\frac{c^2M_\infty}{\det\big(r_1(U_b),r_2(U_b)\big)}\begin{pmatrix}\frac{\mathrm{d}u_b}{\mathrm{d}\tilde{\sigma}_3}+\lambda_2(U_b)\frac{\mathrm{d}v_b}{\mathrm{d}\tilde{\sigma}_3}\\ -\frac{\mathrm{d}u_b}{\mathrm{d}\tilde{\sigma}_3}-\lambda_1(U_b)\frac{\mathrm{d}v_b}{\mathrm{d}\tilde{\sigma}_3}\end{pmatrix}\\
&=-\frac{c\,(1+b_0^2)^2}{2}(1,1)^\top.
\end{aligned}$$

By Lemma 4.4, we have

$$\delta'=\alpha+\beta+O(1)\big(Q^0(\Lambda)+Q^1(\Lambda)\big).$$

Substituting this formula into (6.9) and combining then with Lemma 6.1, we conclude (6.7). ∎

6.2. Λ covers the part of $y = b_\Delta(x)$ but none of $y = \chi_{\Delta x,\vartheta}(x)$. We now consider the wave interactions near the approximate boundary. Suppose that Λ is the diamond centered at $(x_k, y_0(k))$. We denote the two waves entering Λ by α_1 and β_2 that issue from the grid points $(x_{k-1}, y_0(k-1))$ and $(x_{k-1}, y_{-1}(k-1))$, respectively. Let δ_1 be the 1-wave issuing from the grid point $(x_k, y_0(k))$ with U_2 as its above state (see Fig. 6.2). Suppose that the center below the weak waves α_1 and δ_1 is $O_1 = (X, 0)$, between α_1 and the boundary is $O_2 = (\bar{X}, 0)$, and above δ_1 is $O_3 = (\hat{X}, 0)$.

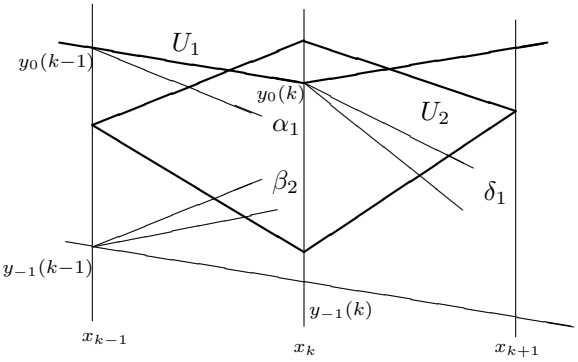

Fig. 6.2. *Reflection at the boundary*

Denote

$$\sigma_0 = \frac{y_0(k)}{x_k - X}, \quad \sigma_1 = \frac{y_{-1}(k-1)}{x_{k-1} - X}, \quad \sigma_2 = \frac{y_0(k-1)}{x_{k-1} - X},$$
$$\bar{\sigma}_0 = \frac{y_0(k-1)}{x_{k-1} - \bar{X}}, \quad \hat{\sigma}_0 = \frac{y_0(k)}{x_k - \hat{X}},$$

and let

$$U_1 = \varpi(\bar{\sigma}_0; O_2), \quad U_2 = \varpi(\hat{\sigma}_0; O_3), \quad U_b = \varpi(\sigma_1; O_1),$$
$$\Delta\sigma = \sigma_0 - \sigma_1, \quad \Delta\tilde{\sigma} = \sigma_2 - \sigma_1, \quad \tilde{x}_0 = |X - \bar{X}|.$$

Then

$$U_1 = \Phi_1(\alpha_1; \varpi(\sigma_2; O_1)), \quad \varpi(\sigma_2; O_1) = \Psi(\Delta\tilde{\sigma}, \sigma_1; \Phi_2(\beta_2; U_b)),$$
$$U_2 = \Phi_1(\delta_1; \varpi(\sigma_0; O_1)), \quad \varpi(\sigma_0; O_1) = \Psi(\Delta\sigma, \sigma_1; U_b).$$

By the construction of the approximate solution, we have

$$\Phi_1(\delta_1; \Psi(\Delta\sigma, \sigma_1; U_b)) \cdot \mathbf{n}_k = \Phi_1(\alpha_1; \Psi(\Delta\tilde{\sigma}, \sigma_1; \Phi_2(\beta_2; U_b))) \cdot \mathbf{n}_{k-1} = 0, \tag{6.10}$$

where $\mathbf{n}_k = (-\sin\theta_k, \cos\theta_k)$ for each fixed $k \geq 0$ is the outer normal vector of the boundary.

To solve equation (6.10), we first have

LEMMA 6.3. $\lim\limits_{M_\infty \to \infty} \dfrac{r_1(U_b) \cdot \mathbf{n}_0}{M_\infty} = c\,(1 + b_0^2)^{-\frac{3}{2}}$.

This can be seen via direct computation by using Lemma 3.6:

$$\lim_{M_\infty\to\infty}\frac{r_1(U_b)\cdot\mathbf{n}_0}{M_\infty}=\lim_{M_\infty\to\infty}\frac{e_1(U_b)}{M_\infty}\lim_{M_\infty\to\infty}\big(\lambda_1(U_b)\sin\theta_0+\cos\theta_0\big)$$
$$=c\,(1+b_0^2)^{-\frac{3}{2}}<\infty.$$

Then we have the following lemma for the existence and estimate of δ_1.

LEMMA 6.4. *Equation* (6.10) *has a unique solution*

$$\delta_1=\delta_1(\alpha_1,\beta_2,\Delta\sigma,\Delta\tilde{\sigma},\omega_k;U_\infty)\in C^2$$

in a neighborhood of $\alpha_1=\beta_2=\Delta\sigma=\Delta\tilde{\sigma}=\omega_k=0$ *and* $U_b=U_\infty$ *such that*

$$\delta_1=\alpha_1+K_{\rm r}\beta_2+K_{\rm b}\omega_k+O(1)|\beta_2||\Delta\sigma|+O(1)|x_k^{-1}-x_{k-1}^{-1}|\tilde{x}_0, \tag{6.11}$$

with

$$\sup_{1<M_\infty<\infty}|K_{\rm b}|<\infty, \tag{6.12}$$

$$K_{\rm r}|_{\{\alpha_1=\beta_2=\omega_k=\Delta\sigma=\tilde{x}_0=0,\ \theta_k=\theta_0\}}=\frac{\cos^2(\theta_0+\theta_{\rm ma})}{\cos^2(\theta_0-\theta_{\rm ma})}, \tag{6.13}$$

which implies that

$$K_{\rm r}=1-4b_0\sqrt{1+b_0^2}\,M_\infty^{-1}+O(1)M_\infty^{-2}+O(1)e^{-m_0M_\infty^2}, \tag{6.14}$$

where the bound of $O(1)$ *depends continuously on* M_∞.

Proof. We divide the proof into four steps.

1. A direct computation gives

$$\frac{\partial}{\partial\delta_1}\Phi_1(\delta_1;\Psi(\Delta\sigma,\sigma_1;U_b))\cdot\mathbf{n}_k\big|_{\{\alpha_1=\beta_2=\omega_k=\Delta\sigma=\Delta\tilde{\sigma}=0,\,\theta_k=\theta_0\}}=r_1(U_b)\cdot\mathbf{n}_0.$$

Then, by Lemma 6.3 and the implicit function theorem, we can find a unique C^2–solution:

$$\delta_1=\delta_1(\alpha_1,\beta_2,\omega_k,\Delta\sigma,\Delta\tilde{\sigma},\sigma_1;U_b)$$

near $(\alpha_1,\beta_2,\Delta\sigma,\Delta\tilde{\sigma};U_b)=(0,0,0,0;U_\infty)$.

Notice that, by a direct computation,

$$\Delta\tilde{\sigma}-\Delta\sigma=\sigma_2-\bar{\sigma}_0+\bar{\sigma}_0-\sigma_0=O(1)\tilde{x}_0|x_k^{-1}-x_{k-1}^{-1}|.$$

Then we have

$$\delta_1=\delta_1'+O(1)|\Delta\tilde{\sigma}-\Delta\sigma|=\delta_1'+O(1)\tilde{x}_0|x_k^{-1}-x_{k-1}^{-1}|,$$

where $\delta_1'(\alpha_1,\beta_2,\omega_k,\Delta\sigma)=\delta_1|_{\Delta\tilde{\sigma}=\Delta\sigma}$ solves the equation:

$$\Phi_1(\delta_1';\Psi(\Delta\sigma,\sigma_1;U_b))\cdot\mathbf{n}_k=\Phi_1(\alpha_1;\Psi(\Delta\sigma,\sigma_1;\Phi_2(\beta_2;U_b)))\cdot\mathbf{n}_{k-1}. \tag{6.15}$$

Let

$$\delta_1''(\alpha_1,\beta_2,\Delta\sigma)=\delta_1'|_{\omega_k=0}=\delta_1(\alpha_1,\beta_2,0,\Delta\sigma,\Delta\sigma,\sigma_1;U_b).$$

Then there exists some $K_{\rm b} \in C^1$ such that

$$\delta_1' - \delta_1'' = K_{\rm b}\omega_k.$$

2. To estimate $K_{\rm b}$, we compute $\frac{\partial \delta_1'}{\partial \omega_k}\Big|_{\{\alpha_1=\beta_2=\omega_k=\Delta\sigma=0,\ \theta_k=\theta_0\}}$. To do this, we take derivative both sides of equation (6.15) with respect to ω_k, and then let $\alpha_1 = \beta_2 = \omega_k = \Delta\sigma = 0$ and $\theta_k = \theta_0$ to obtain

$$r_1(U_b)\cdot \mathbf{n}_0 \left.\frac{\partial \delta_1'}{\partial \omega_k}\right|_{\{\alpha_1=\beta_2=\omega_k=\Delta\sigma=0,\, \theta_k=\theta_0\}} = U_b \cdot (\cos\theta_0, \sin\theta_0).$$

Then we have

$$K_{\rm b}\big|_{\{\alpha_1=\beta_2=\omega_k=\Delta\sigma=0,\ \theta_k=\theta_0\}} = \left.\frac{\partial \delta_2}{\partial \omega_k}\right|_{\{\alpha_1=\beta_1'=\omega_k=\Delta\sigma=\tilde{x}_0=0,\ \theta_k=\theta_0\}} = \frac{u_b\cos\theta_0 + v_b\sin\theta_0}{r_1(U_b)\cdot \mathbf{n}_0},$$

$$\lim_{M_\infty\to\infty} K_{\rm b}|_{\{\beta_2=\omega_k=\Delta\sigma=0,\ \theta_k=\theta_0\}} = \lim_{M_\infty\to\infty} \frac{u_b\cos\theta_0 + v_b\sin\theta_0}{r_1(U_b)\cdot \mathbf{n}_0} = \frac{1}{\sqrt{1+b_0^2}} < \infty,$$

which is uniformly bounded as $M_\infty \to \infty$.

3. Now we are in position to estimate $\delta_1''(\alpha_1, \beta_2, \Delta\sigma)$. Notice that

$$\delta_1''(\alpha_1, 0, \Delta\sigma) = \delta_1''(\alpha_1, 0, 0) = \alpha_1.$$

Then, by (6.2), we have

$$\begin{aligned}\delta_1''(\alpha_1, \beta_2, \Delta\sigma) &= \delta_1''(\alpha_1, 0, \Delta\sigma) + \delta_1''(\alpha_1, \beta_2, 0) - \delta_1''(\alpha_1, 0, 0) + O(1)|\beta_2||\Delta\sigma| \\ &= \delta_1''(\alpha_1, \beta_2, 0) + O(1)|\beta_2||\Delta\sigma|.\end{aligned}$$

Let $\delta_1'''(\alpha_1, \beta_2) = \delta_1''|_{\Delta\sigma=0}$. Then there exists $K_{\rm r} \in C^1$ such that

$$\delta_1'''(\alpha_1, \beta_2) = \delta_1'''(\alpha_1, 0) + K_{\rm r}\beta_2.$$

Note that $\delta_1'''(\alpha_1, \beta_2)$ solves the following equation:

$$\Phi_1(\delta_1'''; U_b)\cdot \mathbf{n}_{k-1} = \Phi(\alpha_1; \Phi_2(\beta_2; U_b))\cdot \mathbf{n}_{k-1}. \tag{6.16}$$

We take derivative both sides of equation (6.16) with respect to β_2 and let $\alpha_1 = \beta_2 = 0$ and $\theta_k = \theta_0$ to obtain

$$(r_1(U_b)\cdot \mathbf{n}_0) \left.\frac{\partial \delta_1'''}{\partial \beta_2}\right|_{\{\alpha_1=\beta_2=0,\ \theta_k=\theta_0\}} = r_2(U_b)\cdot \mathbf{n}_0.$$

It follows that

$$K_{\rm r}|_{\{\alpha_1=\beta_2=0,\, \theta_k=\theta_0\}} = \left.\frac{\partial \delta_3}{\partial \beta_2}\right|_{\{\alpha_1=\beta_2=0,\, \theta_k=\theta_0\}} = \frac{r_2(U_b)\cdot \mathbf{n}_0}{r_1(U_b)\cdot \mathbf{n}_0} = \frac{\cos^2(\theta_0+\theta_{\rm ma})}{\cos^2(\theta_0-\theta_{\rm ma})},$$

which gives the formula for $K_{\rm r}$.

4. Finally, we combine the estimates of δ_1', δ_1'', and δ_1''' with the property that $\delta_1'''(\alpha_1, 0) = \alpha_1$ to conclude the desire result. ∎

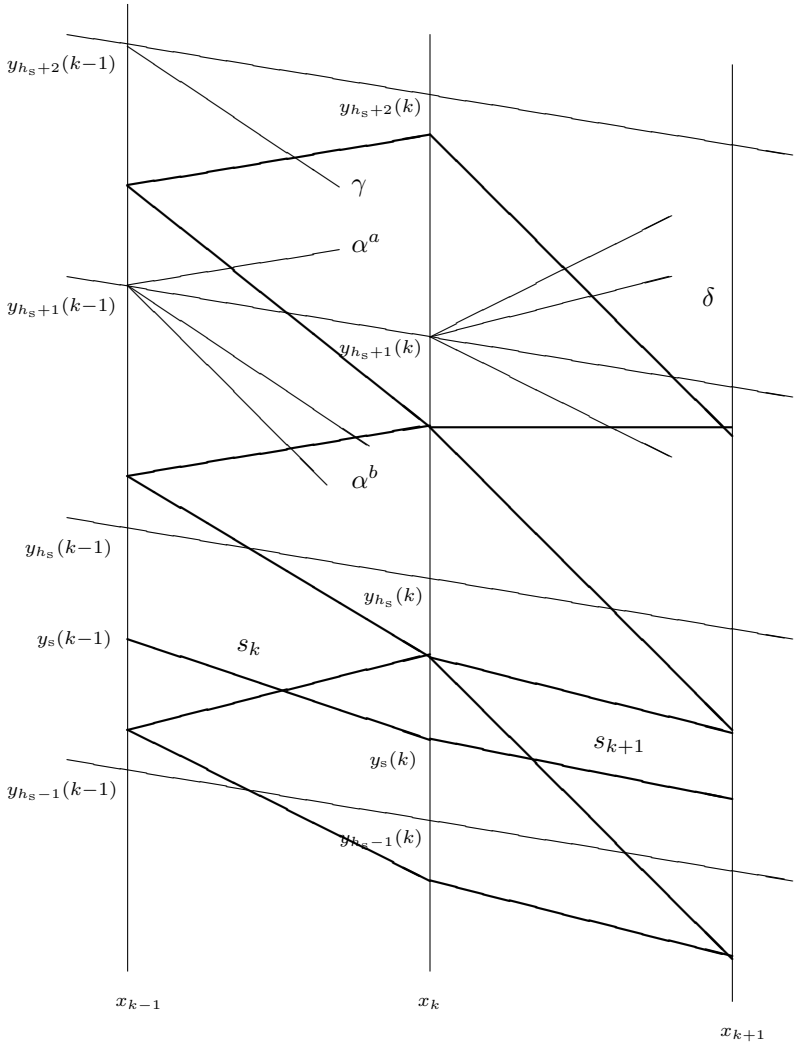

Fig. 6.3. *Interactions between weak waves and the strong wave*

6.3. Λ covers part of $y = \chi_{\Delta x,\vartheta}(x)$ but none of $y = b_\Delta(x)$. We take three diamond simultaneously. As shown in Fig. 6.3, let $\Delta_{k,y_{h_s-1}(k)}$, $\Delta_{k,y_{h_s}(k)}$, and $\Delta_{k,y_{h_s+1}(k)}$ be the diamonds centered in $(x_k, y_{h_s-1}(k))$, $(x_k, y_{h_s}(k))$, and $(x_k, y_{h_s+1}(k))$, respectively. Denote $\Lambda = \Delta_{k,y_{h_s-1}(k)} \cup \Delta_{k,y_{h_s}(k)} \cup \Delta_{k,y_{h_s+1}(k)}$. Let α and γ be the weak waves issuing from $(x_{k-1}, y_{h_s+1}(k-1))$ and $(x_{k-1}, y_{h_s+2}(k-1))$ respectively and entering Λ. We divide α into parts $\alpha^b = (\alpha_1^b, 0)$ and $\alpha^a = (\alpha_1^a, \alpha_2^a)$ with α^b and α^a entering $\Delta_{k,y_{h_s}(k)}$ and $\Delta_{k,y_{h_s+1}(k)}$, respectively. We also assume $\gamma = (\gamma_1, 0)$, and denote δ as the outgoing waves that issue from $(x_k, y_{h_s+1}(k))$.

The center in the region between s_k and the lower edge of α is defined as O_1, in the region between the upper edge of α and the lower edge of γ is defined as O_2, and above the lower edge of γ is denoted as O_3.

Denote the x–coordinate of O_j by X_j^*, $j = 1, 2, 3$, and denote $\tilde{x}_0 := |X_1^* - X_2^*|$ and $\tilde{x}_1 := |X_2^* - X_3^*|$. We also use the coordinates $\sigma = \sigma(x, y) = \frac{y}{x - X_1^*}$, $\bar{\sigma} = \bar{\sigma}(x, y) =$

$\frac{y}{x-X_2^*}$, and $\tilde{\sigma} = \tilde{\sigma}(x,y) = \frac{y}{x-X_3^*}$. Also denote

$$\begin{aligned}
&\sigma_\alpha = \sigma(x_{k-1}, y_{h_{\rm s}+1}(k-1)),\ \sigma_{\rm s}(k-1) = \sigma(x_{k-1}, y_{\rm s}(k-1)),\ \sigma_{\rm s}(k) = \sigma(x_k, y_{\rm s}(k)),\\
&\bar{\sigma}_\alpha = \bar{\sigma}(x_{k-1}, y_{h_{\rm s}+1}(k-1)),\ \bar{\sigma}_{\rm s}(k-1) = \bar{\sigma}(x_{k-1}, y_{\rm s}(k-1)),\ \bar{\sigma}_{\rm s}(k) = \bar{\sigma}(x_k, y_{\rm s}(k)),\\
&\Delta\sigma_\alpha = \sigma_\alpha - \sigma_{\rm s}(k),\ \ \Delta\sigma_{s_k} = \sigma_{\rm s}(k) - \sigma_{\rm s}(k-1),\\
&\Delta\tilde{\sigma}_\gamma = \tilde{\sigma}(x_k, y_{h_{\rm s}+1}(k)) - \tilde{\sigma}(x_{k-1}, y_{h_{\rm s}+2}(k-1)).
\end{aligned}$$

To obtain the estimates of (s_{k+1}, δ), we first consider the following equation:

$$\begin{aligned}
&\Psi(\bar{\sigma}_\alpha - \bar{\sigma}_{\rm s}(k), \bar{\sigma}_{\rm s}(k); \Phi_2(\varepsilon_2; \Theta(s_{k+1})))\\
&= \Phi_1(\alpha_1^b; \Psi(\sigma_\alpha - \sigma_{\rm s}(k-1), \sigma_{\rm s}(k-1); \Theta(s_k))).
\end{aligned} \tag{6.17}$$

With solutions (s_{k+1}, ε_2) of (6.17) and the construction of the approximate solution, we now give the estimates on the weak wave δ.

LEMMA 6.5. *The following asymptotic expansions hold*:

$$\delta_1 = \alpha_1^a + \gamma_1 + O(1)Q(\Lambda), \tag{6.18}$$

$$\delta_2 = \alpha_2^a + K_{\rm w}\alpha_1^b + \mu_{\rm w}\Delta\sigma_{s_k} + O(1)Q(\Lambda), \tag{6.19}$$

$$s_{k+1} = s_k + K_{\rm s}\alpha_1^b + \mu_{\rm s}\Delta\sigma_{s_k} + O(1)\big(|\Delta\sigma_\alpha| + |\Delta\sigma_{s_k}| + |x_k^{-1} - x_{k-1}^{-1}|\big)\tilde{x}_0, \tag{6.20}$$

where

$$\begin{aligned}
Q(\Lambda) =& Q^0(\alpha^a, \gamma) + |\Delta\sigma_\alpha|(|\alpha_1^b| + \tilde{x}_0) + |\Delta\sigma_{s_k}|(|\Delta\sigma_\alpha| + \tilde{x}_0)\\
&+ |\Delta\tilde{\sigma}_\gamma|(|\gamma| + \tilde{x}_1) + |x_k^{-1} - x_{k-1}^{-1}|(\tilde{x}_0 + \tilde{x}_1).
\end{aligned}$$

In addition, for $\alpha^b = \Delta\sigma_\alpha = \Delta\sigma_{s_k} = \tilde{x}_0 = 0$ *and* $s_k = s_0$,

$$K_{\rm w} = \frac{e_1(s_0)}{e_2(s_0)}\frac{\tilde{u}_s(s_0) + \lambda_1(s_0)\tilde{v}_s(s_0)}{\tilde{u}_s(s_0) + \lambda_2(s_0)\tilde{v}_s(s_0)}, \qquad K_{\rm s} = \frac{e_1(s_0)\big(\lambda_2(s_0) - \lambda_1(s_0)\big)}{\tilde{u}_s(s_0) + \lambda_2(s_0)\tilde{v}_s(s_0)}, \tag{6.21}$$

$$\mu_{\rm w} = \frac{\tilde{u}_s(s_0)v_\sigma(s_0) - \tilde{v}_s(s_0)u_\sigma(s_0)}{e_2(s_0)\big(\tilde{u}_s(s_0) + \lambda_2(s_0)\tilde{v}_s(s_0)\big)}, \qquad \mu_{\rm s} = \frac{u_\sigma(s_0) + \lambda_2(s_0)v_\sigma(s_0)}{\tilde{u}_s(s_0) + \lambda_1(s_0)\tilde{v}_s(s_0)}. \tag{6.22}$$

Furthermore, for M_∞ *sufficiently large,*

$$\begin{aligned}
K_{\rm w} &= 1 + 2b_0^{-1}(1 + 2b_0^2)\sqrt{1 + b_0^2}\,M_\infty^{-1} + O(1)M_\infty^{-2} + O(1)e^{-m_0M_\infty^2},\\
K_{\rm s} &= -2b_0^{-1}\sqrt{1 + b_0^2}\,M_\infty^{-1} + O(1)M_\infty^{-2} + O(1)e^{-m_0M_\infty^2},\\
\mu_{\rm w} &= -1 - b_0^{-1}(1 + 2b_0^2)\sqrt{1 + b_0^2}\,M_\infty^{-1} + O(1)M_\infty^{-2} + O(1)e^{-m_0M_\infty^2},\\
\mu_{\rm s} &= b_0^{-1}\sqrt{1 + b_0^2}\,M_\infty^{-1} + O(1)M_\infty^{-2} + O(1)e^{-m_0M_\infty^2},
\end{aligned} \tag{6.23}$$

where $O(1)$ *depends continuously only on* M_∞.

Proof. We divide the proof into four steps.

1. Lemmas 3.8 and 4.5 imply that

$$\lim_{M_\infty\to\infty}\frac{\det(r_2(\Theta(s_0)),\Theta_s(s_0))}{M_\infty^2}=-\lim_{M_\infty\to\infty}\frac{e_2(s_0)}{M_\infty}\frac{\tilde u_s(s_0)+\lambda_2(s_0)\tilde v_s(s_0)}{M_\infty}$$
$$=\frac{c^2b_0}{(1+b_0^2)^3}<\infty.$$

Then, by the implicit function theorem, equation (6.17) admits a unique C^2–solution (δ_2, s_{k+1}) such that

$$\varepsilon_2=\varepsilon_2(\alpha^b,s_k,\sigma_\alpha-\sigma_{\rm s}(k-1),\bar\sigma_\alpha-\bar\sigma_{\rm s}(k),\sigma_{\rm s}(k-1),\bar\sigma_{\rm s}(k)),$$
$$s_{k+1}=s_{k+1}(\alpha^b,s_k,\sigma_\alpha-\sigma_{\rm s}(k-1),\bar\sigma_\alpha-\bar\sigma_{\rm s}(k),\sigma_{\rm s}(k-1),\bar\sigma_{\rm s}(k)).$$

2. Denote

$$\varepsilon_2'=\varepsilon_2'(\alpha^b,s_k,\Delta\sigma_\alpha,\sigma_{\rm s}(k))=\varepsilon_2\big|_{\{\bar\sigma_\alpha-\bar\sigma_{\rm s}(k)=\sigma_\alpha-\sigma_{\rm s}(k-1),\bar\sigma_{\rm s}(k)=\bar\sigma_{\rm s}(k-1)\}},$$
$$s_{k+1}'=s_{k+1}'(\alpha^b,s_k,\Delta\sigma_\alpha,\sigma_{\rm s}(k))=s_{k+1}\big|_{\{\bar\sigma_\alpha-\bar\sigma_{\rm s}(k)=\sigma_\alpha-\sigma_{\rm s}(k-1),\bar\sigma_{\rm s}(k)=\bar\sigma_{\rm s}(k-1)\}}.$$

Then, by a direct computation, we have

$$\varepsilon_2=\varepsilon_2'+O(1)\big(\bar\sigma_{\rm s}(k)-\bar\sigma_{\rm s}(k-1)\big)+O(1)\big(\bar\sigma_\alpha-\bar\sigma_{\rm s}(k)-(\sigma_\alpha-\sigma_{\rm s}(k-1))\big),\tag{6.24}$$
$$s_{k+1}=s_{k+1}'+O(1)\big(\bar\sigma_{\rm s}(k)-\bar\sigma_{\rm s}(k-1)\big)+O(1)\big(\bar\sigma_\alpha-\bar\sigma_{\rm s}(k)-(\sigma_\alpha-\sigma_{\rm s}(k-1))\big),\tag{6.25}$$

where $(\varepsilon_2', s_{k+1}')$ solves the equation:

$$\Psi(\sigma_\alpha-\sigma_{\rm s}(k-1),\bar\sigma_{\rm s}(k-1);\Phi_2(\varepsilon_2';\Theta(s_{k+1}')))$$
$$=\Phi_1(\alpha_1;\Psi(\sigma_\alpha-\sigma_{\rm s}(k-1),\sigma_{\rm s}(k-1);\Theta(s_k))).\tag{6.26}$$

Using the Taylor expansion, we have

$$\varepsilon_2'=K_{\rm w}\alpha^b+\varepsilon_2'',\qquad s_{k+1}'=K_{\rm s}\alpha^b+s_{k+1}'',\tag{6.27}$$

where $(\varepsilon_2'', s_{k+1}'')$ satisfies

$$\Psi(\sigma_\alpha-\sigma_{\rm s}(k-1),\bar\sigma_{\rm s}(k-1);\Phi_2(\varepsilon_2';\Theta(s_{k+1}')))$$
$$=\Psi(\sigma_\alpha-\sigma_{\rm s}(k-1),\sigma_{\rm s}(k-1);\Theta(s_k)).\tag{6.28}$$

Since $\varepsilon_2''|_{\{\sigma_\alpha-\sigma_{\rm s}(k-1)=\tilde x_0=0\}}=0$ and $s_{k+1}''\big|_{\{\sigma_\alpha-\sigma_{\rm s}(k-1)=\tilde x_0=0\}}=0$, then, by (6.2),

$$\varepsilon_2''=O(1)\tilde x_0|\Delta\sigma_\alpha|,\qquad s_{k+1}''=O(1)\tilde x_0|\Delta\sigma_\alpha|.\tag{6.29}$$

Notice that

$$\bar\sigma_{\rm s}(k)-\bar\sigma_{\rm s}(k-1)=\Delta\sigma_{{\rm s}_k}+O(1)\tilde x_0\big(|\Delta\sigma_{{\rm s}_k}|+|x_k^{-1}-x_{k-1}^{-1}|\big),\tag{6.30}$$
$$\bar\sigma_\alpha-\bar\sigma_{\rm s}(k)-(\sigma_\alpha-\sigma_{\rm s}(k-1))$$
$$=\bar\sigma_\alpha-\bar\sigma_{\rm s}(k-1)-(\sigma_\alpha-\sigma_{\rm s}(k-1))+\bar\sigma_{\rm s}(k-1)-\bar\sigma_{\rm s}(k)$$
$$=-\Delta\sigma_{{\rm s}_k}+O(1)\tilde x_0\big(|\Delta\sigma_\alpha|+|\Delta\sigma_{{\rm s}_k}|+|x_k^{-1}-x_{k-1}^{-1}|\big).\tag{6.31}$$

Combining estimates (6.24)–(6.31) together, we obtain the estimates of (ε_2, s_{k+1}):

$$\varepsilon_2 = K_{\rm w}\alpha_1^b + \mu_{\rm w}\Delta\sigma_{{\rm s}_k} + O(1)\tilde{x}_0\big(|\Delta\sigma_\alpha| + |\Delta\sigma_{{\rm s}_k}| + |x_k^{-1} - x_{k-1}^{-1}|\big), \tag{6.32}$$

$$s_{k+1} = s_k + K_{\rm s}\alpha_1^b + \mu_{\rm s}\Delta\sigma_{{\rm s}_k} + O(1)\tilde{x}_0\big(|\Delta\sigma_\alpha| + |\Delta\sigma_{{\rm s}_k}| + |x_k^{-1} - x_{k-1}^{-1}|\big). \tag{6.33}$$

3. To compute coefficients $K_{\rm w}, K_{\rm s}, \mu_{\rm w}$, and $\mu_{\rm s}$, we differentiate (6.16) with respect to α_1^b and $\Delta\sigma_{{\rm s}_k}$, and let $\alpha_1^b = \Delta\sigma_\alpha = \Delta\sigma_{{\rm s}_k} = \tilde{x}_0 = 0$ and $s_k = s_0$ to obtain

$$\begin{aligned} &r_2(\Theta(s_0))K_{\rm w} + \Theta_s(s_0)K_{\rm s} = r_1(\Theta(s_0)),\\ &r_2(\Theta(s_0))\mu_{\rm w} + \Theta_s(s_0)\mu_{\rm s} = \frac{\partial\Psi}{\partial(\Delta\sigma_{{\rm s}_k})}(0, \sigma_{\rm s}(k); \Theta(s_0)). \end{aligned}$$

Cramer's rule implies

$$\begin{aligned} K_{\rm w} &= \frac{\det(r_1(\Theta(s_0)),\ \Theta_s(s_0))}{\det(r_2(\Theta(s_0)),\ \Theta_s(s_0))} = \frac{e_1(s_0)}{e_2(s_0)}\frac{\tilde{u}_s(s_0) + \lambda_1(s_0)\tilde{v}_s(s_0)}{\tilde{u}_s(s_0) + \lambda_2(s_0)\tilde{v}_s(s_0)},\\ K_{\rm s} &= \frac{\det(r_2(\Theta(s_0)),\ r_1(\Theta(s_0)))}{\det(r_2(\Theta(s_0)),\ \Theta_s(s_0))} = \frac{e_1(s_0)(\lambda_2(s_0) - \lambda_1(s_0))}{\tilde{u}_s(s_0) + \lambda_2(s_0)\tilde{v}_s(s_0)}, \end{aligned}$$

and

$$\begin{aligned} \mu_{\rm w} &= \frac{\det(\frac{\partial\Psi}{\partial(\Delta\sigma_{\rm s})}(0, \sigma_{\rm s}(k); \Theta(s_0)), \Theta_s(s_0))}{\det(r_2(\Theta(s_0)), \Theta_s(s_0))},\\ \mu_{\rm s} &= \frac{\det(r_2(\Theta(s_0)), \frac{\partial\Psi}{\partial(\Delta\sigma_{\rm s})}(0, \sigma_{\rm s}(k); \Theta(s_0)))}{\det(r_2(\Theta(s_0)), \Theta_s(s_0))}. \end{aligned}$$

Then, by Lemmas 3.8–3.9 and 4.5, we can estimate $K_{\rm w}$, $K_{\rm s}$, $\mu_{\rm w}$, and $\mu_{\rm s}$ as expected, when M_∞ is sufficiently large.

4. We finally give the estimates of δ. By the construction of the approximate solution, we have

$$\begin{aligned} &\Phi(\delta; \Psi(\Delta\bar{\sigma}_\alpha, \bar{\sigma}_{\rm s}(k); \Theta(s_{k+1})))\\ &= \Psi(\Delta\tilde{\sigma}_\gamma, \tilde{\sigma}(x_{k-1}, y_{h_{\rm s}+2}(k-1)); \Phi(\gamma; \Psi(\bar{\sigma}_\gamma - \bar{\sigma}_\alpha, \bar{\sigma}_\alpha; U_{\rm f}))) \end{aligned} \tag{6.34}$$

with

$$U_{\rm f} = \Phi(\alpha^a; \Psi(\Delta\bar{\sigma}_\alpha, \bar{\sigma}_{\rm s}(k); \Phi_2(\varepsilon_2; \Theta(s_{k+1})))). \tag{6.35}$$

Then, as done in §6.1, we obtain (6.20)–(6.20). ∎

LEMMA 6.6. *For Δx sufficiently small,*

$$\big|\sigma_{\rm s}(k-1) - s_k\big| \geq 6|\Delta\sigma_{{\rm s}_k}|. \tag{6.36}$$

Proof. Notice that

$$s_k = \frac{y_{\rm s}(k) - y_{\rm s}(k-1)}{\Delta x}, \quad \sigma_{\rm s}(k) = \frac{y_{\rm s}(k)}{x_k - X_1^*}.$$

Then, by a direct computation, we have

$$\begin{aligned}
\big|\sigma_{\rm s}(k-1)-s_k\big| &= \Big|\frac{y_{\rm s}(k)-y_{\rm s}(k-1)}{\Delta x}-\sigma_{\rm s}(k-1)\Big| \\
&= \Big|\frac{\sigma_{\rm s}(k)(x_k-X_1^*)-\sigma_{\rm s}(k-1)(x_{k-1}-X_1^*)}{\Delta x}-\sigma_{\rm s}(k-1)\Big| \\
&= \Big|\frac{\sigma_{\rm s}(k)-\sigma_{\rm s}(k-1)}{\Delta x}(x_k-X_1^*)\Big| \\
&\geq 6\big|\sigma_{\rm s}(k)-\sigma_{\rm s}(k-1)\big| \qquad \text{for } \Delta x \text{ sufficiently small.}
\end{aligned}$$

Denote $\theta_{\rm s}(k)=|\sigma_{\rm s}(k-1)-s_k|$, which measures the angle between the leading shock s_k and the line passing through $(x_{k-1},y_{\rm s}(k-1))$ and the center of s_k. Moreover, we have the following estimate for $\theta_{\rm s}(k)$.

LEMMA 6.7. *For M_∞ sufficiently large and Δx sufficiently small,*

$$\theta_{\rm s}(k)-\theta_{\rm s}(k+1)\geq|\Delta\sigma_{s_k}|+\widetilde{K}_{\rm s}|\alpha_1^b|-C\tilde{x}_0\big(|\Delta\sigma_\alpha|+|\Delta\sigma_{s_k}|+\big|x_k^{-1}-x_{k-1}^{-1}\big|\big), \tag{6.37}$$

where $k\geq 0$, $|\widetilde{K}_{\rm s}|=|K_{\rm s}|$ is given by Lemma 6.5, and constant $C>0$ is independent of M_∞ and Δx.

Proof. The proof is divided into two subcases.

1. $\sigma_{\rm s}(k-1)<s_k$ so that $\sigma_{\rm s}(k-1)<\sigma_{\rm s}(k)$.

- If $s_{k+1}>\sigma_{\rm s}(k)$, then, by Lemma 6.5,

$$\begin{aligned}
\theta_{\rm s}(k)-\theta_{\rm s}(k+1) &= s_k-\sigma_{\rm s}(k-1)-\big(s_{k+1}-\sigma_{\rm s}(k)\big) \\
&= (1-\mu_{\rm s})\Delta\sigma_{s_k}-K_{\rm s}\alpha_1^b \\
&\quad +O(1)\tilde{x}_0\big(|\Delta\sigma_\alpha|+|\Delta\sigma_{s_k}|+\big|x_k^{-1}-x_{k-1}^{-1}\big|\big) \\
&\geq |\Delta\sigma_{s_k}|-K_{\rm s}\alpha_1^b-C\tilde{x}_0\big(|\Delta\sigma_\alpha|+|\Delta\sigma_{s_k}|+\big|x_k^{-1}-x_{k-1}^{-1}\big|\big).
\end{aligned}$$

- If $s_{k+1}<\sigma_{\rm s}(k)$, then, by Lemmas 6.5–6.6,

$$\begin{aligned}
\theta_{\rm s}(k)-\theta_{\rm s}(k+1) &= s_k-\sigma_{\rm s}(k-1)-\big(s_{k+1}-\sigma_{\rm s}(k)\big) \\
&= 2\big(s_k-\sigma_{\rm s}(k-1)\big)+s_{k+1}-\sigma_{\rm s}(k)-\big(s_k-\sigma_{\rm s}(k-1)\big) \\
&\geq (11+\mu_{\rm s})|\Delta\sigma_{s_k}|+K_{\rm s}\alpha_1^b \\
&\quad -C\tilde{x}_0\big(|\Delta\sigma_\alpha|+|\Delta\sigma_{s_k}|+\big|x_k^{-1}-x_{k-1}^{-1}\big|\big) \\
&\geq |\Delta\sigma_{s_k}|+K_{\rm s}\alpha_1^b-C\tilde{x}_0\big(|\Delta\sigma_\alpha|+|\Delta\sigma_{s_k}|+\big|x_k^{-1}-x_{k-1}^{-1}\big|\big).
\end{aligned}$$

2. $s_k<\sigma_{\rm s}(k-1)$ so that $\sigma_{\rm s}(k)<\sigma_{\rm s}(k-1)$.

- If $s_{k+1}>\sigma_{\rm s}(k)$, then, by Lemmas 6.5–6.6,

$$\begin{aligned}
\theta_{\rm s}(k)-\theta_{\rm s}(k+1) &= s_k-\sigma_{\rm s}(k)-\big(s_{k+1}-\sigma_{\rm s}(k+1)\big) \\
&= 2\big(s_k-\sigma_{\rm s}(k)\big)+s_k-\sigma_{\rm s}(k-1)-\big(s_{k+1}-\sigma_{\rm s}(k)\big) \\
&\geq (\mu_{\rm s}+11)|\Delta\sigma_{s_k}|-K_{\rm s}\alpha_1^b \\
&\quad -C\tilde{x}_0\big(|\Delta\sigma_\alpha|+|\Delta\sigma_{s_k}|+\big|x_k^{-1}-x_{k-1}^{-1}\big|\big) \\
&\geq |\Delta\sigma_{s_k}|-K_{\rm s}\alpha_1^b-C\tilde{x}_0\big(|\Delta\sigma_\alpha|+|\Delta\sigma_{s_k}|+\big|x_k^{-1}-x_{k-1}^{-1}\big|\big).
\end{aligned}$$

- If $s_{k+1} < \sigma_{\rm s}(k)$, then, by Lemma 6.5,

$$\begin{aligned}\theta_{\rm s}(k) - \theta_{\rm s}(k+1) &= \sigma_{\rm s}(k-1) - s_k - \sigma_{\rm s}(k) + s_{k+1}\\ &= (1-\mu_{\rm s})|\Delta\sigma_{{\rm s}_k}| + K_{\rm s}\alpha_1^b\\ &\quad + O(1)\tilde{x}_0\big(|\Delta\sigma_\alpha| + |\Delta\sigma_{{\rm s}_k}| + \big|x_k^{-1} - x_{k-1}^{-1}\big|\big)\\ &\geq |\Delta\sigma_{{\rm s}_k}| - K_{\rm s}\alpha_1^b - C\tilde{x}_0\big(|\Delta\sigma_\alpha| + |\Delta\sigma_{{\rm s}_k}| + \big|x_k^{-1} - x_{k-1}^{-1}\big|\big).\end{aligned}$$

In the above estimates, we have used the fact that $\mu_{\rm s} \in (-1, 0)$ for M_∞ sufficiently large. This completes the proof. ∎

7. The Glimm-Type Functional and the Convergence of the Approximate Solutions. In this section, we first apply the difference scheme and the local interaction estimates obtained in §6 above to construct a suitable Glimm-type functional for the approximate solutions, and then prove its monotonicity so that the total variation of the approximate solutions in y is uniformly bounded in x. Thus, we first state a lemma which is important to prove the monotonicity of the Glimm-type functional.

LEMMA 7.1. *Let $K_{\rm r}$, $K_{\rm w}$, $K_{\rm s}$ and $\mu_{\rm w}$ be given by Lemmas 6.4–6.6. Then, for M_∞ sufficiently large,*

$$|K_{\rm r}|\big(|K_{\rm w}| + |K_{\rm s}||\mu_{\rm w}|\big) < 1. \tag{7.1}$$

Moreover, there exist positive constants K_1, K_2, and K_3 such that

$$|K_{\rm r}| - K_2 < 0, \quad K_2|\mu_{\rm w}| - K_3 < 0, \quad K_2|K_{\rm w}| + K_3|K_{\rm s}| - 1 < 0. \tag{7.2}$$

Proof. By (6.15) in Lemma 6.4 and (6.24) in Lemma 6.5, we obtain that, for M_∞ sufficiently large,

$$\begin{aligned}&|K_{\rm r}|\big(|K_{\rm w}| + |K_{\rm s}||\mu_{\rm w}|\big)\\ &\quad = 1 - 2b_0^{-2}(1+b_0^2)(8b_0^4 + 2b_0^2 + 1)\, M_\infty^{-1} + O(1)M_\infty^{-2} + O(1)e^{-m_0 M_\infty^2}.\end{aligned} \tag{7.3}$$

Note that the second term on the right-hand side of (7.3) is negative. Thus, we can choose M_∞ sufficiently large such that (7.1) holds. It follows that there exists a constant $K_2 > 0$ such that

$$K_2 > |K_{\rm r}|, \qquad K_2\big(|K_{\rm w}| + |K_{\rm s}||\mu_{\rm w}|\big) < 1,$$

which leads to

$$K_2|K_{\rm s}||\mu_{\rm w}| < 1 - K_2|K_{\rm w}|.$$

Then we can choose another constant $K_3 > 0$ such that

$$K_2|\mu_{\rm w}| - |K_3| < 0, \qquad |K_3||K_{\rm s}| + K_2|K_{\rm w}| - 1 < 0.$$

This completes the proof. ∎

We now turn to the construction of the Glimm functional and study its properties. Let J be a space-like mesh curve connecting the mesh points. Denote Γ_J as the set of the corner points with A_k lying in J^+, *i.e.*,

$$\Gamma_J = \{A_k \,:\, A_k = (x_k, b_k), A_k \in J^+, k \geq 0\}.$$

Then we define the following Glimm-type functional:

DEFINITION 7.2 (Weighted total variation). *Denote*

$$L_0^{(i)}(J) := \sum \Big\{ |\alpha_i| \,:\, \alpha_i \text{ is } i\text{-weak wave crossing } J \Big\}, \quad i = 1, 2,$$

$$L_1(J) := \sum \Big\{ (1 + |b_k|)|\omega(A_k)| \,:\, A_k \in \Gamma_J \Big\},$$

$$L_{\rm s}(J) := \theta_{\rm s}(J) \qquad \text{for } \theta_{\rm s}(J) \text{ as } \theta_{\rm s}(k) \text{ in Lemma 6.7 when s crosses } J,$$

$$L_{\rm c}(J) := \sum \Big\{ \big|X^*_{\alpha+} - X^*_{\alpha-}\big|\big(1 + x_\alpha^{-1}\big) \,:\, \alpha \text{ is 1-wave crossing } J \text{ from } x = x_\alpha \Big\},$$

where $x_\alpha \in \{x_k \,:\, k \geq 0\}$, *and* $X^*_{\alpha\pm}$ *denote the limits of* X^* *on the right and left of* α. *Then the weighted total variation is defined as*

$$L(J) := L_0^{(1)}(J) + K_2 L_0^{(2)}(J) + K_1 L_1(J) + K_3 L_{\rm s}(J) + K_4 L_{\rm c}(J), \tag{7.4}$$

where K_1, K_2, *and* K_3 *are given as in Lemma 7.1, and* $K_4 > 0$ *is a constant to be specified later.*

Next, we turn to the construction of quadratic terms for the total interaction potential.

DEFINITION 7.3 (Total interaction potential). *Denote*

$$Q_0(J) := \sum \Big\{ |\alpha_i||\beta_j| \,:\, \alpha_i \text{ and } \beta_j \text{ are weak waves that cross } J \text{ and approach} \Big\},$$

$$Q_1(J) := \sum \Big\{ |\alpha|(\sigma_\alpha - \sigma_*) \,:\, \alpha \text{ is 1-weak wave crossing } J \Big\},$$

$$Q_2(J) := \sum \Big\{ |\alpha|(\sigma^* - \sigma_\alpha) \,:\, \alpha \text{ is 2-weak wave crossing } J \Big\},$$

$$Q_{\rm c}(J) := \sum \Big\{ |X^*_{\alpha+} - X^*_{\alpha-}|(\sigma^{\rm c}_\alpha(J) - \sigma_*) \,:\, \alpha \text{ is } i\text{-weak wave crossing } J,\ i = 1, 2 \Big\},$$

$$Q_{\rm ce}(J) := \sum \Big\{ |X^*_{\alpha+} - X^*_{\alpha-}||X^*_{\beta+} - X^*_{\beta-}| \,:\, \alpha \text{ and } \beta \text{ are weak waves crossing } J \Big\},$$

$$Q^{(j)}_{\rm wc}(J) := \sum \Big\{ |\beta_j||X^*_{\alpha+} - X^*_{\alpha-}| \,:\, \alpha \text{ is 1-weak wave above } j\text{-weak wave } \beta_j \text{ on } J \Big\},$$

for $j = 1, 2$, *where* $X^*_{\alpha\pm}$ *denote the right and left limits of* X^* *of* α, σ_α *is the* σ*-coordinate of the grid point where* α *issues, and* $\sigma^{\rm c}_\alpha(J)$ *is the* σ*-coordinate of the grid point where the center of the self-similar solution passing through* J *changes from* $X^*_{\alpha-}$ *to* $X^*_{\alpha+}$. *In addition, denote*

$$\sigma^* := b_0 + C_1 \sum (1 + |b_k|)|\omega_k|, \qquad \sigma_* = s_0 - \varrho,$$

where s_0 *is the speed of the leading shock-front for the background problem,* b_0 *is the unperturbed boundary slope,* ϱ *and* C_1 *are positive constants chosen so that* $Q_1(J)$, $Q_2(J)$, *and* $Q_{\rm c}(J)$ *are nonnegative. Note that* ϱ *and* $(1+|b_k|)|\omega_k|$ *are chosen small so that the largeness of* M_∞ *implies the smallness of* $s_0 - b_0$ *that leads to the smallness of* $\sigma^* - \sigma_*$. *The summation in* $Q^{(j)}_{\rm wc}(J)$ *is taken over for all couples of weak waves* (α, β_j), $j = 1, 2$.

Then the total interaction potential is defined as

$$Q(J) := \sum_{i=0,1,2,{\rm c}} Q_i(J) + \sum_{i=1,2} K^{(i)}_{\rm wc}\, Q^{(i)}_{\rm wc}(J) + K_{\rm ce}\, Q_{\rm ce}(J), \tag{7.5}$$

where $K^{(i)}_{\rm wc}, i = 1, 2$, *and* $K_{\rm ce}$ *are positive constants to be specified later.*

Finally, we give the definition of the Glimm-type functional.

DEFINITION 7.4 (Glimm-type functional).

$$F(J) := L(J) + KQ(J), \tag{7.6}$$

where $K > 0$ is a constant to be given below.

Let

$$E_{\Delta x,\vartheta}(\Lambda) = \begin{cases} Q(\Lambda) & \text{(defined in §6.1)},\\ |\beta_2| + (1+|b_k|)|\omega_k| & \text{(defined in §6.2)},\\ Q(\Lambda) + |\alpha_1^b| + |\Delta\sigma_{\mathrm{s}_k}| & \text{(defined in §6.3)}. \end{cases} \tag{7.7}$$

With the notations given above, we now prove the decreasing property of our functional $F(J)$ by specifying constants K, K_i with $i = 1,2,3,4$, K_{ce}, and $K_{\mathrm{wc}}^{(i)}$ with $i = 1,2$.

We first have the main global interaction estimate as stated below:

PROPOSITION 7.5. *Suppose that M_∞ is sufficiently large and $\sum_{k=0}^{\infty}\big(1+|b_k|\big)|\omega_k|$ is sufficiently small. Let I and J be a pair of space-like mesh curves with $I < J$, and let Λ be the diamond between I and J. Then there exist positive constants η, K_i with $i = 1,2,3,4$, $K_{\mathrm{wc}}^{(i)}$ with $i = 1,2$, K_{ce}, and K such that, if $F(I) < \eta$,*

$$F(J) \le F(I) - \frac{1}{4}E_{\Delta x,\vartheta}(\Lambda), \tag{7.8}$$

where $E_{\Delta x,\vartheta}(\Lambda)$ is given by (7.7).

Proof. By induction, on the mesh curves, it suffices to consider the case that J is an immediate successor to I with only one diamond Λ between I and J. Let $I = I_0 \cup I'$ and $J = I_0 \cup J'$. As in §4, we also divide our analysis into three cases depending on the location of the diamond. From now on, we denote $C > 0$ a universal constant depending only on the system, which may be different at each occurrence.

Case 1. *Λ lies between the cone boundary and the leading shock-front.* We now consider the case as in Lemma 6.2. Notice that

$$\begin{aligned} &\big(L_0^{(1)} + K_2 L_0^{(2)}\big)(J) - \big(L_0^{(1)} + K_2 L_0^{(2)}\big)(I) \le C\,Q(\Lambda),\\ &\big(K_1 L_1 + K_3 L_{\mathrm{s}}\big)(J) - \big(K_1 L_1 + K_3 L_{\mathrm{s}}\big)(I) = 0, \end{aligned}$$

and

$$\begin{aligned} L_{\mathrm{c}}(J) - L_{\mathrm{c}}(I) &= |X - \tilde{X}|(1 + x_k^{-1}) - |X - \bar{X}|(1 + x_{k-1}^{-1}) - |\bar{X} - \tilde{X}|(1 + x_{k-1}^{-1})\\ &\le -|x_k^{-1} - x_{k-1}^{-1}|(\tilde{x}_0 + \tilde{x}_1). \end{aligned}$$

Then

$$L(J) - L(I) \le C\,Q(\Lambda) - K_4|x_k^{-1} - x_{k-1}^{-1}|(\tilde{x}_0 + \tilde{x}_1).$$

For Q, we have

$$
\begin{aligned}
&Q_0(J)-Q_0(I)\le C\,L(I)Q(\Lambda)-Q^0(\Lambda),\\
&\big(Q_1+Q_2\big)(J)-\big(Q_1+Q_2\big)(I)\le C(\sigma^*-\sigma_*)Q(\Lambda)+C\tilde{x}_0|\beta|-|\Delta\sigma||\alpha|,\\
&Q_{\rm c}(J)-Q_{\rm c}(I)\le C\tilde{x}_0\tilde{x}_1-|\Delta\sigma|\tilde{x}_0,\\
&\sum_{i=1,2}Q^{(i)}_{\rm wc}(J)-\sum_{i=1,2}Q^{(i)}_{\rm wc}(I)\le-\sum_{i=1,2}K^{(i)}_{\rm wc}|\beta_i|\tilde{x}_0,\\
&Q_{\rm ce}(J)-Q_{\rm ce}(I)\le-\tilde{x}_0\tilde{x}_1.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
Q(J)-Q(I)\le&-\big(1-C(L(I)+\sigma^*-\sigma_*)\big)Q(\Lambda)+|x_k^{-1}-x_{k-1}^{-1}|\tilde{x}_0\\
&-\big(K^{(1)}_{\rm wc}-C\big)|\beta_1|\tilde{x}_0-\big(K^{(2)}_{\rm wc}-C\big)|\beta_2|\tilde{x}_0-\big(K_{\rm ce}-C\big)\tilde{x}_0\tilde{x}_1.
\end{aligned}
$$

Therefore, it follows that

$$
\begin{aligned}
F(J)-F(I)\le&-\Big\{K\big(1-C(L(I)+\sigma^*-\sigma_*)\big)-C\Big\}Q(\Lambda)\\
&-(K_4-K)\,|x_k^{-1}-x_{k-1}^{-1}|\tilde{x}_0-K\big(K^{(1)}_{\rm wc}-C\big)|\beta_1|\tilde{x}_0\\
&-K\big(K^{(2)}_{\rm wc}-C\big)|\beta_2|\tilde{x}_0-K\big(K_{\rm ce}-C\big)\tilde{x}_0\tilde{x}_1\\
\le&-\frac{1}{4}Q(\Lambda),
\end{aligned}
$$

provided that $L(I)$ and $\sigma^*-\sigma_*$ are small enough, and $K^{(i)}_{\rm wc}$ with $i=1,2$, $K_{\rm ce}$, and $K>K_4$ are sufficiently large.

Case 2. Λ *covers part of the cone boundary but not the leading shock-front.* We consider only the case as given in Lemma 6.4. Since $|\bar{X}-\hat{X}|=O(1)|b_k||\omega_k|$ by Lemma 5.1, a direct computation yields that

$$
\begin{aligned}
&L_0^{(1)}(J)-L_0^{(1)}(I)\le|K_{\rm r}||\beta_2|+|K_{\rm b}||\omega_k|+C|\beta_2||\Delta\sigma|+C|x_k^{-1}-x_{k-1}^{-1}|\tilde{x}_0,\\
&L_0^{(2)}(J)-L_0^{(2)}(I)=-|\beta_2|,\\
&L_1(J)-L_1(I)=-(1+|b_k|)|\omega_k|,\\
&L_{\rm c}(J)-L_{\rm c}(I)\le-|x_k^{-1}-x_{k-1}^{-1}|\tilde{x}_0+C(1+x_k^{-1})|b_k||\omega_k|,\\
&L_{\rm S}(J)-L_{\rm S}(I)=0.
\end{aligned}
$$

Then

$$
\begin{aligned}
L(J)-L(I)\le&-(K_2-|K_{\rm r}|)|\beta_2|-(K_1-|K_{\rm b}|)|\omega_k|-(K_4-C)|x_k^{-1}-x_{k-1}^{-1}|\tilde{x}_0\\
&-\big(K_1-K_4C\,(1+x_k^{-1})\big)|b_k||\omega_k|+C\,|\beta_2||\Delta\sigma|.
\end{aligned}
$$

For Q, we have

$$
\begin{aligned}
&Q_0(J)-Q_0(I)\le C\big(|\beta_2|+|\omega_k|+|\beta_2||\Delta\sigma|+|x_k^{-1}-x_{k-1}^{-1}|\tilde{x}_0\big)L(I),\\
&Q_1(J)-Q_1(I)\le C\big(|\beta_2|+|\omega_k|+|\beta_2||\Delta\sigma|+|x_k^{-1}-x_{k-1}^{-1}|\tilde{x}_0\big)(\sigma^*-\sigma_*)\\
&\qquad\qquad +CL(I)|b_k||\omega_k|,\\
&Q_2(J)-Q_2(I)\le -|\beta_2||\Delta\sigma|,\\
&Q_{\rm c}(J)-Q_{\rm c}(I)\le C(\sigma^*-\sigma_*)|b_k||\omega_k|+C\,L(I)|x_k^{-1}-x_{k-1}^{-1}|\tilde{x}_0,\\
&\big(K_{\rm wc}^{(1)}Q_{\rm wc}^{(1)}+K_{\rm wc}^{(2)}Q_{\rm wc}^{(2)}\big)(J)-\big(K_{\rm wc}^{(1)}Q_{\rm wc}^{(1)}+K_{\rm wc}^{(2)}Q_{\rm wc}^{(2)}\big)(I)\le 0,\\
&Q_{\rm ce}(J)-Q_{\rm ce}(I)\le C\,L(I)|b_k||\omega_k|.
\end{aligned}
$$

Thus, we obtain the following estimate for Q:

$$
\begin{aligned}
Q(J)-Q(I)\le &-\big(1-C(L(I)+\sigma^*-\sigma_*)\big)|\beta_2||\Delta\sigma|\\
&+C\big(L(I)+\sigma^*-\sigma_*\big)\big(|\beta_2|+|\omega_k|+|b_k||\omega_k|+|x_k^{-1}-x_{k-1}^{-1}|\tilde{x}_0\big).
\end{aligned}
$$

Finally, we obtain the estimate for $F(J)$:

$$
\begin{aligned}
F(J)-F(I)\le &-\big(K_2-|K_{\rm r}|-KC(L(I)+\sigma^*-\sigma_*)\big)|\beta_2|\\
&-\big(K_1-|K_{\rm b}|-KC(L(I)+\sigma^*-\sigma_*)\big)|\omega_k|\\
&-\big(K_1-K_4C(1+x_k^{-1})-KC(L(I)+\sigma^*-\sigma_*)\big)|b_k||\omega_k|\\
&-\big(K_4-C-KC(L(I)+\sigma^*-\sigma_*)\big)|x_k^{-1}-x_{k-1}^{-1}|\tilde{x}_0\\
&-\big(K(1-C(L(I)+\sigma^*-\sigma_*))-C\big)|\beta_2||\Delta\sigma|.
\end{aligned}
$$

Using Lemma 7.1, choosing K sufficiently large, and letting $L(I)$ and $\sigma^*-\sigma_*$ sufficiently small, we have

$$
F(J)-F(I)\le -\frac{1}{4}\big(|\beta_2|+(1+|b_k|)|\omega_k|\big).
$$

Case 3. Λ *covers a part of the leading shock-front.* By Lemma 6.5, we have

$$
\begin{aligned}
&L_0^{(1)}(J)-L_0^{(1)}(I)\le -|\alpha_1^b|+CQ(\Lambda),\\
&L_0^{(2)}(J)-L_0^{(2)}(I)\le |K_{\rm w}||\alpha_1^b|+|\mu_{\rm s}||\Delta\sigma_{{\rm s}_k}|+CQ(\Lambda),\\
&L_1(J)-L_1(I)=0,\\
&L_{\rm s}(J)-L_{\rm s}(I)\le C\big(|\Delta\sigma_\alpha|+|\Delta\sigma_{{\rm s}_k}|+|x_k^{-1}-x_{k-1}^{-1}|\big)\tilde{x}_0-|\Delta\sigma_{{\rm s}_k}|-|K_{\rm s}||\alpha_1^b|,\\
&L_{\rm c}(J)-L_{\rm c}(I)=|X_1^*-X_3^*|(1+x_k^{-1})-|X_1^*-X_2^*|(1+x_{k-1}^{-1})\\
&\qquad\qquad -|X_2^*-X_3^*|(1+x_{k-1}^{-1})\\
&\qquad\qquad \le -|x_k^{-1}-x_{k-1}^{-1}|(\tilde{x}_0+\tilde{x}_1).
\end{aligned}
$$

Then we combine the estimates for $L_0^{(1)}, L_0^{(2)}$, $L_1, L_{\rm s}$, and $L_{\rm c}$ to obtain

$$
\begin{aligned}
L(J)-L(I)\le &-\big(1-K_2|K_{\rm w}|+K_3|K_{\rm s}|\big)|\alpha_1^b|-\big(K_3-K_2|\mu_{\rm w}|\big)|\Delta\sigma_{{\rm s}_k}|\\
&-\big(K_4-K_3C\big)|x_k^{-1}-x_{k-1}^{-1}|\tilde{x}_0-K_4|x_k^{-1}-x_{k-1}^{-1}|\tilde{x}_1\\
&+CQ(\Lambda)+C\big(|\Delta\sigma_\alpha|+|\Delta\sigma_{{\rm s}_k}|\big)\tilde{x}_0.
\end{aligned}
$$

Next, we estimate Q:

$$
\begin{aligned}
&Q_0(J)-Q_0(I)\le -Q^0(\alpha^a,\gamma)+\big(|\mu_{\rm w}||\Delta\sigma_{{\rm s}_k}|+|K_{\rm w}||\alpha_1^b|-|\alpha_1^b|\big)L(I)+CL(I)Q(\Lambda),\\
&Q_1(J)-Q_1(I)\le -|\gamma||\Delta\tilde\sigma_\gamma|-|\alpha_1^b||\Delta\sigma_\alpha|+C|\alpha_1^a|\tilde x_1+C(\sigma^*-\sigma_*)Q(\Lambda),\\
&Q_2(J)-Q_2(I)\le (\sigma^*-\sigma_*)\big(|K_{\rm w}||\alpha_1^b|+|\mu_{\rm w}||\Delta\sigma_{{\rm s}_k}|\big)+C|\alpha_2^a|\tilde x_1+C(\sigma^*-\sigma_*)Q(\Lambda),\\
&Q_{\rm c}(J)-Q_{\rm c}(I)\le -|\Delta\sigma_\alpha|\tilde x_0-|\Delta\tilde\sigma_\gamma|\tilde x_1+C\tilde x_0\tilde x_1,\\
&\big(K_{\rm wc}^{(1)}Q_{\rm wc}^{(1)}+K_{\rm wc}^{(2)}Q_{\rm wc}^{(2)}\big)(J)-\big(K_{\rm wc}^{(1)}Q_{\rm wc}^{(1)}+K_{\rm wc}^{(2)}Q_{\rm wc}^{(2)}\big)(I)\le -K_{\rm wc}^{(1)}|\alpha_1^a|\tilde x_1-K_{\rm wc}^{(2)}|\alpha_2^a|\tilde x_1,\\
&Q_{\rm ce}(J)-Q_{\rm ce}(I)\le -\tilde x_0\tilde x_1.
\end{aligned}
$$

Then

$$
\begin{aligned}
&Q(J)-Q(I)\\
&\le -\big(1-C(L(I)+\sigma^*-\sigma_*)\big)Q(\Lambda)-\big(K_{\rm wc}^{(1)}-C\big)|\alpha_1^a|\tilde x_1-\big(K_{\rm wc}^{(2)}-C\big)|\alpha_2^a|\tilde x_1\\
&\quad-\big(K_{\rm ce}-C\big)\tilde x_0\tilde x_1+\big(|K_{\rm w}|(\sigma^*-\sigma_*)+(|K_{\rm w}|-1)L(I)\big)|\alpha_1^b|\\
&\quad+(|\mu_{\rm w}|+1)(L(I)+\sigma^*-\sigma_*)|\Delta\sigma_{{\rm s}_k}|+|x_k^{-1}-x_{k-1}^{-1}|(\tilde x_0+\tilde x_1).
\end{aligned}
$$

Finally, we combine the estimates of L and Q to obtain

$$
\begin{aligned}
F(J)-F(I)\le &-\Big\{K\big(1-C(L(I)+\sigma^*-\sigma_*)\big)-C\Big\}Q(\Lambda)\\
&-\Big\{1-K_2|K_{\rm w}|+K_3|K_{\rm s}|-K\big(|K_{\rm w}|(\sigma^*-\sigma_*)+(|K_{\rm w}|-1)L(I)\big)\Big\}|\alpha_1^b|\\
&-\Big\{K_3-K_2|\mu_{\rm w}|-K(1+|\mu_{\rm w}|)\big(L(I)+\sigma^*-\sigma_*\big)\Big\}|\Delta\sigma_{{\rm s}_k}|\\
&-\big(K_4-K_3C-K\big)|x_k^{-1}-x_{k-1}^{-1}|\tilde x_0-\big(K_4-K\big)|x_k^{-1}-x_{k-1}^{-1}|\tilde x_1\\
&-K\Big\{\big(K_{\rm wc}^{(1)}-C\big)|\alpha_1^a|\tilde x_1+\big(K_{\rm wc}^{(2)}-C\big)|\alpha_2^a|\tilde x_1+\big(K_{\rm ce}-C\big)\tilde x_0\tilde x_1\Big\}.
\end{aligned}
$$

Using (6.2), choosing $K>K_4$, $K_{\rm wc}^{(1)},K_{\rm wc}^{(2)}$, and $K_{\rm ce}$ sufficiently large, and taking $L(I)$ and $\sigma^*-\sigma_*$ sufficiently small, we have

$$
F(J)-F(I)\le -\frac14\big(Q(\Lambda)+|\alpha_1^b|+|\Delta\sigma_{{\rm s}_k}|\big).
$$

Now we choose an appropriate constant ϱ such that, for any 1-wave α after interaction, $\sigma_\alpha\ge s_0-\varrho$. By (6.21), we have

$$
|s_{k+1}-s_k|\le |K_{\rm s}\alpha_1|+|\mu_{\rm s}\Delta\sigma_{{\rm s}_k}|+C(|\Delta\sigma_\alpha|+|\Delta\sigma_{{\rm s}_k}|)\tilde x_0.
$$

Then the monotonicity of the Glimm functional implies that there exists a constant $C_2>0$ such that

$$
\sum_{k\ge0}\big|s_{k+1}-s_k\big|\le C_2\sum_{J>I}\big(F(I)-F(J)\big)\le C_2F(0),
$$

which leads to

$$
s_0-C_2F(0)\le s_{k+1}\le s_0+C_2F(0).
$$

Since σ_α satisfies $\sigma_\alpha\ge s_{k+1}-CF(0)$, there exists a positive constant C_3 such that $\sigma_\alpha\ge s_0-C_3F(0)$. Then we choose $\varrho=C_3F(0)$. ∎

For any weak wave α, denote $\sigma(x_\alpha, y_\alpha)$ as the corresponding self-similar variable for point (x_α, y_α) from where the weak wave α is issued, with $x_\alpha \in \{x_k : k \geq 0\}$. Denote $U_{\Delta x,\vartheta}(x, y)$ as the approximate solution. Then we have

PROPOSITION 7.6. *For M_∞ sufficiently large and $\sum_{k=0}^{\infty}(1+|b_k|)|\omega_k|$ sufficiently small, if*

$$\lambda_1(U_{\Delta x,\vartheta}(x_\alpha-,\cdot)) < \sigma(x_\alpha-,\cdot) < \lambda_2(U_{\Delta x,\vartheta}(x_\alpha-,\cdot)), \tag{7.9}$$

then

$$\lambda_1(U_{\Delta x,\vartheta}(x_\alpha+,\cdot)) < \sigma(x_\alpha+,\cdot) < \lambda_2(U_{\Delta x,\vartheta}(x_\alpha+,\cdot)). \tag{7.10}$$

Proof. For any two mesh curves I and J satisfying $I < J$, we prove the lemma by induction.

Since $|b_k - b_{k-1}| = O(1)|\omega_k|$, we assume that

$$|b_k - b_0| \leq \sum_{j=1}^{k} |b_j - b_{j-1}| = O(1)\sum_{j=0}^{k} |\omega_k| \leq C\big(F(0) - F(I)\big).$$

Then, using assumption (7.9), Lemma 6.4, and Proposition 7.5, we have

$$\begin{aligned} |b_{k+1} - b_0| &\leq |b_{k+1} - b_k| + |b_k - b_0| \\ &\leq C|\omega_{k+1}| + C\big(F(0) - F(I)\big) \\ &\leq C\big(F(I) - F(J) + F(0) - F(I)\big) \\ &\leq CF(0). \end{aligned} \tag{7.11}$$

Similarly, for s_k, assume that

$$\big|s_k - s_0\big| \leq C\big(F(0) - F(I)\big).$$

Then, using assumption (7.9), Lemma 6.5, and Proposition 7.5 again, we have

$$\begin{aligned} |s_{k+1} - s_0| &\leq |s_{k+1} - s_k| + |s_k - s_0| \\ &\leq C\big(|\alpha_1^b| + |\Delta\sigma_{s_k}| + \tilde{x}_0\big) + C\big(F(0) - F(I)\big) \\ &\leq C\big(F(I) - F(J) + F(0) - F(I)\big) \\ &\leq CF(0). \end{aligned} \tag{7.12}$$

Since

$$s_{k+1} - CF(0) < \sigma(x_\alpha+,\cdot) < b_{k+1} + CF(0),$$

it follows from (7.11)–(7.12) that

$$s_0 - CF(0) < \sigma(x_\alpha+,\cdot) < b_0 + CF(0). \tag{7.13}$$

On the other hand, since

$$\lambda_1(U_{\Delta x,\vartheta}(x_\alpha+,\cdot)) - \lambda_1(\Theta(s_{k+1})) = O(1)F(0),$$

it follows from Lemma 6.5 and Proposition 7.5 that

$$\lambda_1(\Theta(s_{k+1}))-\lambda_1(\Theta(s_0))=O(1)(s_{k+1}-s_0)=O(1)F(0).$$

Then

$$\begin{aligned}&\lambda_1(U_{\Delta x,\vartheta}(x_\alpha+,\cdot))-\lambda_1(\Theta(s_0))\\&=\lambda_1(U_{\Delta x,\vartheta}(x_\alpha+,\cdot))-\lambda_1(\Theta(s_{k+1}))+\lambda_1(\Theta(s_{k+1}))-\lambda_1(\Theta(s_0))\\&=O(1)F(0),\end{aligned}$$

which leads to

$$\begin{aligned}\lambda_1(U_{\Delta x,\vartheta}(x_\alpha+,\cdot))=b_0-(1+b_0^2)^{\frac{3}{2}}M_\infty^{-1}+O(1)F(0)\\+O(1)M_\infty^{-2}+O(1)e^{-m_0M_\infty^2}.\end{aligned}\tag{7.14}$$

Similarly, for $\lambda_2(U_{\Delta x,\vartheta})$, we have

$$\begin{aligned}\lambda_2(U_{\Delta x,\vartheta}(x_\alpha+,\cdot))=b_0+(1+b_0^2)^{\frac{3}{2}}M_\infty^{-1}+O(1)F(0)\\+O(1)M_\infty^{-2}+O(1)e^{-m_0M_\infty^2}.\end{aligned}\tag{7.15}$$

Finally, by (7.13)–(7.15), we obtain

$$\begin{aligned}&\lambda_1(U_{\Delta x,\vartheta}(x_\alpha+,\cdot))-\sigma(x_\alpha+,\cdot)\\&<-(1+b_0^2)^{\frac{3}{2}}M_\infty^{-1}+C\big(F(0)+M_\infty^{-2}+e^{-m_0M_\infty^2}\big),\end{aligned}\tag{7.16}$$

$$\begin{aligned}&\sigma(x_\alpha+,\cdot)-\lambda_2(U_{\Delta x,\vartheta}(x_\alpha+,\cdot))\\&<-(1+b_0^2)^{\frac{3}{2}}M_\infty^{-1}+C\big(F(0)+M_\infty^{-2}+e^{-m_0M_\infty^2}\big).\end{aligned}\tag{7.17}$$

Since M_∞ is sufficiently large and $F(0)=O(1)\sum_{k=0}^{\infty}(1+|b_k|)|\omega_k|$ is sufficiently small, then, by (7.16)–(7.17), we obtain

$$\lambda_1\big(U_{\Delta x,\vartheta}(x_\alpha+,\cdot)\big)<\sigma\big(x_\alpha+,\cdot\big)<\lambda_2\big(U_{\Delta x,\vartheta}(x_\alpha+,\cdot)\big).$$

This complete the proof. ∎

Then, applying Proposition 7.5 and following the methods as done in [9, 29], we conclude

THEOREM 7.7. *Under assumptions* $(\mathbf{H_1})$–$(\mathbf{H_2})$, *if* M_∞ *is sufficiently large and* $\sum_{k\geq 0}(1+|b_k|)|\omega_k|$ *is sufficiently small, then, for any* $\vartheta\in\Pi_{k=0}^{\infty}(-1,1)$ *and every* $\Delta x>0$, *the modified Glimm scheme developed above defines a sequence of global approximate solutions* $U_{\Delta x,\vartheta}(x,y)$ *such that*

$$\sup_{x>0}T.V.\big\{U_{\Delta x,\vartheta}(x,y)\,:\,-\infty<y<b_\Delta(x)\big\}<\infty,\tag{7.18}$$

$$\int_{-\infty}^{0}\big|U_{\Delta x,\vartheta}(x_1,y+b_\Delta(x_1))-U_{\Delta x,\vartheta}(x_2,y+b_\Delta(x_2))\big|\,\mathrm{d}y\leq C_4|x_1-x_2|,\tag{7.19}$$

where $C_4>0$ *is independent of* $U_{\Delta x,\vartheta}$, Δx, *and* ϑ.

Denote

$$s_{\Delta x,\vartheta}(x)=\mathbf{1}_{(k\Delta x,(k+1)\Delta x)}s_k\qquad\text{for } k\geq 0,\tag{7.20}$$

where $\mathbf{1}_A$ stands for the characteristic function on set A. Then, by direct computation, we have

$$\chi_{\Delta x,\vartheta}(x) = \int_0^x s_{\Delta x,\vartheta}(\tau)\,\mathrm{d}\tau. \tag{7.21}$$

Moreover, by Lemma 6.5 and Proposition 7.5, we have

COROLLARY 7.8. *There exists a constant $C_5 > 0$ independent of $U_{\Delta x,\vartheta}$, Δx, and ϑ such that*

$$T.V.\,\{s_{\Delta x,\vartheta}(x) \,:\, x \in [0,\infty)\} \le C_5. \tag{7.22}$$

Once the uniform boundedness of the total variation of the approximate solutions $U_{\Delta x,\vartheta}$ is obtained, then, by Proposition 7.5 and Corollary 7.8, the convergence of $U_{\Delta x,\vartheta}$ follows. We can prove that its limit U_ϑ is actually an entropy solution of problem (1.1)–(1.7). This can be summarized as the following theorem whose proof is standard and similar to [9, 17, 29], so we omit the details here.

THEOREM 7.9. *Let assumptions $(\mathbf{H_1})$–$(\mathbf{H_2})$ hold. Assume that*

$$\int_0^\infty (1 + |b(x)|)\,\mathrm{d}\mu(x) < \tilde{\varepsilon}, \tag{7.23}$$

where $\mu(x) = T.V.\big\{b'_+(\tau) \,:\, \tau \in [0,x)\big\}$. Then there is a null set $\mathcal{N}$ such that, if M_∞ is sufficiently large and $\tilde{\varepsilon} > 0$ sufficiently small, for each $\vartheta \in (\Pi_{k=0}^\infty(-1,1)\setminus\mathcal{N})$, there exist both a subsequence $\{\Delta_i\}_{i=0}^\infty \subset \{\Delta x\}$ of mesh sizes with $\Delta_i \to 0$ as $i \to \infty$ and a pair of functions $U_\vartheta(x,y) \in O_{\tilde{\varepsilon}}(\Gamma(b_0,u_\infty))$ and $\chi_\vartheta(x)$ with $\chi_\vartheta(0) = 0$ such that

(i) *$U_{\Delta_i,\vartheta}(x,\cdot)$ converges to $U_\vartheta(x,\cdot)$ in $L^1(-\infty,b(x))$ for every $x > 0$ as $i \to \infty$, and U_ϑ is a global entropy solution of problem (1.1)–(1.7);*

(ii) *$s_{\Delta_i,\vartheta}(x)$ converges to $s_\vartheta(x) \in BV([0,\infty))$ with $|s_\vartheta(x) - s_0| < C\tilde{\varepsilon}$;*

(iii) *$\chi_{\Delta_i,\vartheta}(x)$ converges to $\chi_\vartheta(x)$ uniformly in any bounded x–interval such that*

$$\chi_\vartheta(x) = \int_0^x s_\vartheta(\tau)\,\mathrm{d}\tau, \tag{7.24}$$

and $\chi_\vartheta(x) < b(x)$ for any $x > 0$, where $C > 0$ is a constant depending only on the system.

8. Asymptotic Behavior of Global Entropy Solutions. To understand the asymptotic behavior of global entropy solutions $U_\vartheta(x,y)$, we need further estimates of the approximate solutions $U_{\Delta_i,\vartheta}(x,y)$.

LEMMA 8.1. *There exists a constant M_1 independent of $U_{\Delta x,\vartheta}$, Δx, and ϑ such that*

$$\sum_\Lambda E_{\Delta x,\vartheta}(\Lambda) < M_1, \tag{8.1}$$

for $E_{\Delta x,\vartheta}(\Lambda)$ given as in (7.7).

Proof. By Proposition 7.5, for any interaction diamond $\Lambda \subset \{(k-1)\Delta x \le x \le (k+1)\Delta x\}$, $k \ge 1$, we have

$$\sum_\Lambda E_{\Delta x,\vartheta}(\Lambda) \le 4\sum_\Lambda \big(F(I) - F(J)\big) \le 4F(0).$$

Then estimate (8.1) follows by choosing $M_1 = 4F(0) + 1$. ∎

For any $\tau > 0$, let $\mathcal{L}_{j,\vartheta}(\tau-)$, $j = 1, 2$, be the total variation of j-weak waves in U_ϑ crossing line $x = \tau$, and let $\mathcal{L}_{j,\Delta x,\vartheta}(\tau-)$, $j = 1, 2$, be the total variation of j-weak waves in $U_{\Delta x,\vartheta}$ crossing line $x = \tau$. In addition, denote by $\mathcal{C}_\vartheta(\tau-)$ the total variation for the centers in U_ϑ when the self-similar lines cross line $x = \tau$, and let $\mathcal{C}_{\Delta x,\vartheta}(\tau-)$ be the total variation of the center changes in $U_{\Delta x,\vartheta}$ when the self-similar lines cross line $x = \tau$. Then we have

LEMMA 8.2. *As $x \to \infty$,*

$$\sum_{j=1}^{2} \mathcal{L}_{j,\vartheta}(x-) + \mathcal{C}_\vartheta(x-) \longrightarrow 0.$$

Proof. Let $U_{\Delta_i,\vartheta}(x, y)$ be a sequence of the approximate solutions stated in Theorem 7.9, and let the corresponding term $E_{\Delta_i,\vartheta}(\Lambda)$ be defined in (7.7). As in [17], denote by $\mathrm{d}E_{\Delta_i,\vartheta}$ the measures of the assigning quantities $E_{\Delta_i,\vartheta}(\Lambda)$ of the centers of Λ. Then, by Lemma 8.1, we can select a subsequence (still denoted as $\mathrm{d}E_{\Delta_i,\vartheta}$) such that

$$\mathrm{d}E_{\Delta_i,\vartheta} \to \mathrm{d}E_\vartheta \qquad \text{as } \Delta_i \to 0,$$

with $E_\vartheta(\Lambda) < \infty$.

Therefore, for $\hat{\varepsilon} > 0$ sufficiently small, we can choose $x_{\hat{\varepsilon}} > 0$ (independent of $U_{\Delta_i,\vartheta}$), Δ_i, and ϑ such that

$$\sum_{k>[x_{\hat{\varepsilon}}/\Delta x]} E_{\Delta_i,\vartheta}(\Lambda_{k,n}) < \hat{\varepsilon}.$$

Let $X^1_{\hat{\varepsilon}} = (x_{\hat{\varepsilon}}, \chi_{\Delta_i,\vartheta}(x_{\hat{\varepsilon}}))$ and $X^2_{\hat{\varepsilon}} = (x_{\hat{\varepsilon}}, b_{\Delta_i}(x_{\hat{\varepsilon}}))$ be the two points lying in the approximate 1-shock $y = \chi_{\Delta_i,\vartheta}(x)$ and the approximate cone boundary Γ_{Δ_i}, respectively. Let $\chi^j_{\Delta_i,\vartheta}$ be the approximate j–generalized characteristic issuing from $X^j_{\hat{\varepsilon}}$ for $j = 1, 2$, respectively. According to the construction of the approximate solution, there exist constants $\hat{M}_j > 0$ for $j = 1, 2$, independent of $U_{\Delta_i,\vartheta}$, Δ_i, and ϑ, such that

$$\big|\chi^j_{\Delta_i,\vartheta}(x_1) - \chi^j_{\Delta_i,\vartheta}(x_2)\big| \le \hat{M}_j\big(|x_1 - x_2| + \Delta_i\big) \qquad \text{for } x_1, x_2 > x_{\hat{\varepsilon}}.$$

Then we can choose a subsequence (still denoted by) Δ_i such that

$$\chi^j_{\Delta_i,\vartheta}(x) \to \chi^j_\vartheta(x) \qquad \text{as } \Delta_i \to 0,$$

for some $\chi^j_\vartheta \in \mathrm{Lip}$ with $(\chi^j_\vartheta)'$ bounded.

Let the two characteristics $\chi^1_\vartheta(x)$ and $\chi^2_\vartheta(x)$ intersect with the cone boundary $\partial\Omega$ and shock-front $y = \chi_\vartheta(x)$ at points $(t^1_{\hat{\varepsilon}}, \chi^1(t^1_{\hat{\varepsilon}}))$ and $(t^2_{\hat{\varepsilon}}, \chi^2(t^2_{\hat{\varepsilon}}))$ for some $t^1_{\hat{\varepsilon}}$ and $t^2_{\hat{\varepsilon}}$, respectively. Then, as in [18], we apply the approximate conservation law to the domain below $\chi^1_{\Delta_i,\vartheta}$ and above $\chi^1_{\Delta_i,\vartheta}$, and use Lemma 8.1 to obtain

$$\mathcal{L}_{j,\Delta_i,\vartheta}(x-) \le C \sum_{k>[x_{\hat{\varepsilon}}/\Delta x]} E_{\Delta_i,\vartheta}(\Lambda_{k,n}) \le C\hat{\varepsilon},$$

$$\mathcal{C}_{\Delta x,\vartheta}(x-) \le C \sum_{k>[x_{\hat{\varepsilon}}/\Delta x]} (1 + |b_k|)|\omega_k| \le C\hat{\varepsilon},$$

for $j = 1, 2$, and $x > t^1_{\hat{\varepsilon}} + t^2_{\hat{\varepsilon}}$, where the bound of $O(1)$ is independent of $U_{\Delta x,\vartheta}$, Δx, and ϑ. These lead to

$$\mathcal{L}_{j,\vartheta}(x-) = O(1)\hat{\varepsilon}, \qquad \mathcal{C}_\vartheta(x-) = O(1)\hat{\varepsilon},$$

for $j = 1, 2$, and $x > t^1_{\hat{\varepsilon}} + t^2_{\hat{\varepsilon}}$. This completes the proof. ∎

Denote

$$X^*_\infty = \lim_{x\to\infty} X^*(x, b(x)). \tag{8.2}$$

THEOREM 8.3. *Let U_ϑ be the entropy solution of problem (1.4)–(1.7) given by Theorem 7.9. Denote $s_\infty = \lim_{x\to\infty} s_\vartheta(x)$ and $b'_\infty = \lim_{x\to\infty} b'_+(x)$. Then*

$$\lim_{x\to\infty} \sup \big\{ |U_\vartheta(x,y) - \varpi(\sigma_\infty; O_\infty)| \,:\, \chi_\vartheta(x) < y < b(x) \big\} = 0, \tag{8.3}$$

where $\varpi(\sigma_\infty; O_\infty)$ is the state of the self-similar solutions with $\sigma_\infty = \frac{y}{x - X^*_\infty}$ and $O_\infty = (X^*_\infty, 0)$ as its self-similar variable and center respectively, and satisfies

$$\varpi(s_\infty; O_\infty) = \Theta(s_\infty), \qquad \varpi(b'_\infty; O_\infty)\cdot(-b'_\infty, 1) = 0 \tag{8.4}$$

with $\Theta(s)$ as the state connected to state U_∞ by the 1-shock of speed s.

Proof. From the construction of the approximate solution, there exists a state $\varpi(\sigma_k; O_k)$ such that $U_{\Delta_i,\vartheta}(x,y) = \varpi(\sigma_k; O_k)$ for some $k \geq 1$ with $\sigma_k = \frac{y}{x - X^*_k}$ and $O_k = (X^*_k, 0)$. Let $x \in [(l-1)\Delta x, l\Delta x)$ for some $l \geq 1$, and let

$$X^*_l = x_{l-1} - \frac{b_{l-1}}{b_l - b_{l-1}}\Delta x.$$

Then, for every $x > 0$,

$$\begin{aligned} \big|U_{\Delta_i,\vartheta}(x,y) - \varpi(\sigma_l; O_l)\big| &= |\varpi(\sigma_k; O_k) - \varpi(\sigma_l; O_l)| \\ &\leq C\Big(\sum_{j=1,2} \mathcal{L}_{j,\Delta_i,\vartheta}(x-) + \mathcal{C}_{\Delta_i,\vartheta}(x-)\Big), \end{aligned}$$

where $C > 0$ is independent of $U_{\Delta_i,\vartheta}$, Δ_i, and ϑ.

On the other hand, for every $x > 0$

$$|\varpi(\sigma_l; O_l) - \varpi(\sigma_\infty; O_\infty)| \leq C\, \mathcal{C}_{\Delta x,\vartheta}(x-),$$

where $C > 0$ is independent of $U_{\Delta_i,\vartheta}$, Δ_i, and ϑ. Thus, for every $x > 0$, we obtain

$$\begin{aligned} &|\varpi(s_{\Delta_i;\vartheta}; O_l) - \Theta(s_{\Delta_i,\vartheta}(x))| + |\varpi(b'_{\Delta_i}(x); O_l)\cdot(-b'_{\Delta_i}, 1)| \\ &+ |U_{\Delta_i,\vartheta}(x,\cdot) - \varpi(\sigma_\infty; O_\infty)| \\ &\leq \sup_{\chi_{\Delta_i,\vartheta}(x) < y < b_{\Delta_i}(x)} |U_{\Delta_i,\vartheta}(x,y) - \varpi(\sigma_l; O_l)| \\ &\quad + \sup_{\chi_{\Delta_i,\vartheta}(x) < y < b_{\Delta_i}(x)} |\varpi(\sigma_l; O_l) - \varpi(\sigma_\infty; O_\infty)| \\ &\leq C\Big(\sum_{j=1,2} \mathcal{L}_{j,\Delta_i,\vartheta}(x-) + \mathcal{C}_{\Delta_i,\vartheta}(x-)\Big). \end{aligned}$$

By Theorem 7.9, letting $\Delta_i \to 0$, we have

$$\begin{aligned} &|\varpi(s_\vartheta; O_l) - \Theta(s_\vartheta(x))| + |\varpi(b'_+(x); O_l)\cdot(-b'_+, 1)| \\ &+ \sup_{\chi_\vartheta(x) < y < b(x)} |U_\vartheta(x,y) - \varpi(\sigma_\infty; O_\infty)| \\ &\leq C\Big(\sum_{j=1,2} \mathcal{L}_{j,\vartheta}(x-) + \mathcal{C}_\vartheta(x-)\Big) \qquad \text{for every } x > 0, \end{aligned}$$

which leads to the desire result by using Lemma 8.2. ∎

Appendix A. Proof of Lemma 2.1. In this appendix, we give a proof of Lemma 2.1 by showing the fact that system (2.1) is genuinely nonlinear for $u > c$. We first introduce some notations for the computational convenience.

Denote $q = \sqrt{u^2+v^2}$, $M = \frac{q}{c}$, $\theta = \arctan\frac{v}{u}$, and $\lambda_j = \lambda_j(U)$, $j = 1, 2$. Then the eigenvalues can be rewritten as

$$\lambda_j = \tan(\theta + (-1)^j \theta_{\rm ma}) \qquad \text{for } j = 1, 2, \tag{A.1}$$

where $\theta_{\rm ma}$ is the Mach angle:

$$\theta_{\rm ma} := \arctan\Big(\frac{1}{\sqrt{M^2-1}}\Big). \tag{A.2}$$

Moreover, $\theta_{\rm ma} = \arcsin(\frac{1}{M}) \in (0, \frac{\pi}{2})$ for supersonic flow. Then we have

LEMMA A.1. *If $u > c$, then*

$$\cos(\theta + (-1)^j \theta_{\rm ma}) = \frac{u\sqrt{M^2-1} + (-1)^{j+1} v}{cM^2} > 0 \qquad \textit{for } j = 1, 2. \tag{A.3}$$

Proof. By direct computation, we obtain the first equality:

$$cM^2 \cos(\theta + (-1)^j \theta_{\rm ma}) = u\sqrt{M^2-1} + (-1)^{j+1} v \qquad \text{for } j = 1, 2.$$

Since $\big(u\sqrt{q^2-c^2}\big)^2 - c^2 v^2 = (u^2-c^2)q^2 > 0$. Then

$$u\sqrt{M^2-1} > |u|.$$

This completes the proof. ∎

LEMMA A.2. *If $u > c$, then*

$$\frac{\partial q}{\partial u} = \cos\theta, \quad \frac{\partial q}{\partial v} = \sin\theta, \quad \frac{\partial \theta}{\partial u} = -\frac{\sin\theta}{q}, \quad \frac{\partial \theta}{\partial v} = \frac{\cos\theta}{q}, \quad \frac{\partial \theta_{\rm ma}}{\partial q} = -\frac{1}{cM\sqrt{M^2-1}}.$$

Proof. We prove only for the last identity above, since the proofs for the others are similar. From (A.2), we have

$$\cos\theta_{\rm ma} = \frac{\sqrt{M^2-1}}{M}.$$

Therefore, we have

$$\frac{\partial \theta_{\rm ma}}{\partial q} = -\frac{M}{(M^2-1)^{\frac{3}{2}}} \cos^2\theta_{\rm ma} = -\frac{1}{cM\sqrt{M^2-1}}.$$

This completes the proof. ∎

LEMMA A.3. *If $u > c$, then*

$$\frac{\partial \lambda_j}{\partial \theta} = \sec^2(\theta + (-1)^j \theta_{\rm ma}), \quad \frac{\partial \lambda_j}{\partial q} = (-1)^{j+1} \frac{1}{cM\sqrt{M^2-1}} \sec^2(\theta + (-1)^j \theta_{\rm ma})$$

for $j = 1, 2$.

LEMMA A.4. *If $u > c$, then*

$$\frac{\partial \lambda_j}{\partial u} = \frac{(-1)^j}{c\sqrt{M^2-1}} \sin(\theta + (-1)^j \theta_{\mathrm{ma}}) \sec^2(\theta + (-1)^j \theta_{\mathrm{ma}}),$$
$$\frac{\partial \lambda_j}{\partial v} = \frac{(-1)^{j+1}}{c\sqrt{M^2-1}} \cos(\theta + (-1)^j \theta_{\mathrm{ma}}) \sec^2(\theta + (-1)^j \theta_{\mathrm{ma}}) \qquad \textit{for } j = 1, 2.$$

Proof. For $j = 1$, from Lemmas A.2–A.3, we have

$$\begin{aligned}
\frac{\partial \lambda_1}{\partial u} &= \frac{\partial \lambda_1}{\partial \theta}\frac{\partial \theta}{\partial u} + \frac{\partial \lambda_1}{\partial q}\frac{\partial q}{\partial u} \\
&= -\sec^2(\theta - \theta_{\mathrm{ma}}) \frac{\sin\theta}{q} + \frac{1}{cM\sqrt{M^2-1}} \sec^2(\theta - \theta_{\mathrm{ma}}) \cos\theta \\
&= -\frac{1}{c\sqrt{M^2-1}} \sin(\theta - \theta_{\mathrm{ma}}) \sec^2(\theta - \theta_{\mathrm{ma}}), \\
\frac{\partial \lambda_1}{\partial v} &= \frac{\partial \lambda_1}{\partial \theta}\frac{\partial \theta}{\partial v} + \frac{\partial \lambda_1}{\partial q}\frac{\partial q}{\partial v} \\
&= \sec^2(\theta - \theta_{\mathrm{ma}}) \frac{\cos\theta}{cM} + \frac{1}{cM\sqrt{M^2-1}} \sec^2(\theta - \theta_{\mathrm{ma}}) \sin\theta \\
&= \frac{1}{c\sqrt{M^2-1}} \cos(\theta - \theta_{\mathrm{ma}}) \sec^2(\theta - \theta_{\mathrm{ma}}).
\end{aligned}$$

The case for $j = 2$ can be carried out in the same way. This completes the proof. ∎

LEMMA A.5. *For $u > c$,*

$$(\frac{\partial \lambda_j}{\partial u}, \frac{\partial \lambda_j}{\partial v}) \cdot (-\lambda_j, 1) = \frac{\sec^3(\theta + (-1)^j \theta_{\mathrm{ma}})}{c\sqrt{M^2-1}} \qquad \textit{for } j = 1, 2. \tag{A.4}$$

Proof. We consider only for the case that $j = 1$, since it is similar for $j = 2$. By (A.4)–(A.4) and Lemma A.4, we know that

$$\begin{aligned}
(\frac{\partial \lambda_1}{\partial u}, \frac{\partial \lambda_1}{\partial v}) \cdot (-\lambda_1, 1) &= \frac{\sec^3(\theta - \theta_{\mathrm{ma}}) \sin^2(\theta - \theta_{\mathrm{ma}})}{c\sqrt{M^2-1}} + \frac{\sec^2(\theta - \theta_{\mathrm{ma}}) \cos(\theta - \theta_{\mathrm{ma}})}{c\sqrt{M^2-1}} \\
&= \frac{\sec^3(\theta - \theta_{\mathrm{ma}})}{c\sqrt{M^2-1}}.
\end{aligned}$$

This completes the proof. ∎

From Lemma A.1, we know that system (2.1)–(2.2) is genuinely-nonlinear for $u > c$.

Then, according to Lemmas A.1 and A.5, we have the following property that leads to the proof of Lemma 2.1.

LEMMA A.6. *For $u > c$,*

$$e_j(U) = \sqrt{q^2 - c^2} \cos^3\left(\theta + (-1)^j \theta_{\mathrm{ma}}\right) > 0 \qquad \textit{for } j = 1, 2. \tag{A.5}$$

Acknowledgments. The authors would like to thank the anonymous referees for helpful comments.

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